NONCOVERAGE FOR DISTINCT ODD MODULI WITH AT MOST THREE PRIME DIVISORS

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Abstract. We prove that a finite family of congruence classes with pairwise distinct odd moduli greater than one cannot cover the integers if each modulus has at most three distinct prime divisors. The prime powers, residue classes, and total number of primes occurring across the family are unrestricted. A normalized distortion measure and a conditional convex comparison reduce the problem to a count of one-prime and two-prime cofactors. Twenty-eight factorial moments close under a positive recurrence; exact rational arithmetic through prime 997 and a bound for all larger primes give total charge less than $0.937076 < 15/16$. We also prove that the three-divisor restriction is needed only for moduli whose largest prime divisor is at most 10^9. Thus every hypothetical odd distinct covering has a modulus with at least four distinct prime divisors, all at most 10^9. Two standalone exact-arithmetic implementations accompany the proof. A finite rational alternative to the integral tail bound underlies an end-to-end Lean 4 formalization of the first noncoverage theorem.

1. Introduction

A *covering system* is a finite family of congruence classes whose union is $\mathbb{Z}$. It is *distinct* if its moduli are pairwise distinct, and *odd* if all its moduli are odd. Modulus 1 is excluded throughout. For an integer $m > 1$, let $\omega(m)$ denote its number of distinct prime divisors, and let $P^+(m)$ denote its largest prime divisor. We establish the following restriction on the individual moduli.

Theorem 1.1. *Let $\mathcal{F} = \{a_i \pmod{m_i} : 1 \le i \le s\}$ be a finite family with pairwise distinct odd moduli $m_i > 1$. If $\omega(m_i) \le 3$ for every i, then $\mathcal{F}$ does not cover $\mathbb{Z}$.*

The theorem allows arbitrary exponents and arbitrary residue classes. There is no bound on s, on the primes themselves, or on $\omega(\operatorname{lcm}(m_1, \dots, m_s))$.

Date: 9 September 2026; revised 15 September 2026.

2020 *Mathematics Subject Classification.* Primary 11B25; Secondary 11A07, 05D40.

Key words and phrases. Erdős–Selfridge odd covering problem, Erdős Problem 7, covering systems, prime divisors, prime powers, distortion method, convex comparison, exact computation.

Version 1.0. DOI: 10.5281/zenodo.22760638.

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In particular, different moduli may use different triples of primes. A stronger statement confines a necessary exception to a fixed range of prime divisors.

Theorem 1.2. *Let $\mathcal{F}$ be as in Theorem 1.1. Suppose that*

$$P^+(m_i) \le 10^9 \quad \Longrightarrow \quad \omega(m_i) \le 3.$$

Then $\mathcal{F}$ does not cover $\mathbb{Z}$.

Corollary 1.3. *Every finite odd distinct covering system has a modulus m such that*

$$\omega(m) \ge 4, \qquad P^+(m) \le 10^9.$$

Proof. The assertion is the contrapositive of Theorem 1.2. □

The cutoff in the corollary concerns prime divisors, not the size of m. Neither theorem requires minimality, disjointness, or a complete divisor list. Both concern finite families. They provide necessary conditions for an odd distinct covering and do not resolve the unrestricted odd covering problem.

1.1. **Context.** Erdős introduced distinct covering systems in 1950 [6]. Balister's survey [1] describes the classical questions and the development of the distortion method. The number of prime divisors of an individual modulus is a classical direction: his 1973 survey records Selfridge's squarefree two-factor observation and a corresponding three-factor expectation [7, p. 125]. The present hypothesis permits arbitrary powers of the small primes as well as of the large ones.

The principal squarefree comparison is the theorem of Balister, Bollobás, Morris, Sahasrabudhe and Tiba [3], excluding odd distinct coverings with squarefree moduli. Their final paragraph requires squarefreeness only at primes at most 73. Their theorem allows moduli with arbitrarily many prime divisors; Theorem 1.1 instead allows unrestricted powers at every prime. These hypothesis classes are incomparable.

Hough's solution of the minimum-modulus problem [8] and the restricted-divisibility theorem of Hough and Nielsen [9] developed the probabilistic approach to distinct coverings. The latter forces a modulus divisible by 2 or 3. The distortion method of Balister et al. [2, 3] controls the mass of forbidden fibres while preserving earlier marginals. Our local measure belongs to this framework. The convex comparison used here is a finite conditional application of the classical extremal property of comonotone dependence; see [12]. We prove the required version in full.

The author's seven-prime, eight-prime and nine-prime results [14, 15, 16] concern the primes in the least common multiple of the entire family. They give successive necessary lower bounds of seven, eight and nine for that total prime support, with arbitrary prime-power exponents. The present argument bounds the support of each individual modulus and places no bound on the total support. None of those rank theorems is used as a lemma below.

Recent work also treats different restrictions. Bispels et al. [4] study a variant permitting one odd modulus to repeat. Mian and Siddique [11] give a Lean formalization excluding odd distinct coverings with least common multiple at most 10^4. These settings differ from the arbitrary-power, three-divisor condition of Theorem 1.1.

1.2. **Proof outline and computational scope.** We pass to the finite Chinese-remainder product resolving the family and avoid all pure prime-power classes. A normalized prime-by-prime distortion then gives uniform bounds for prefix probabilities conditional on the entire past. A backward convex comparison replaces the actual, possibly dependent coordinates by independent auxiliary heights K_p.

At a prime q, a modulus satisfying $\omega(m) \le 3$ has at most two prime divisors left after its q-power is removed. The completed cofactor count is therefore

$$N_q = \sum_{p<q} K_p + \sum_{p<r<q} K_p K_r.$$

Adjoining a height K updates (N, Y) to $(N + (1 + Y)K, Y + K)$, where Y is the sum of the old heights. Its binomial moments of total order at most six form a positive, closed system of 28 quantities. An exact low-state calculation supplies the initial charges. Directed rational rounding and a polynomial tail estimate give

$$\sum_q b_q < \frac{937076}{10^6} < \frac{15}{16}. \tag{1.1}$$

Here b_q bounds the probability of the mixed union ending at q under one constructed law. Positive mass consequently remains uncovered.

Sections 2–4 prove the reduction to these charges. Sections 5–7 give the complete finite recurrence, its exact certificate, and the bound for every remaining prime. Section 8 proves Theorem 1.2 using a second-moment continuation. Appendix A specifies both arithmetic implementations. Appendix B gives a finite rational alternative to the integral tail, compares its certificate with Section 6, and describes the formal verification of Theorem 1.1. The mathematical proof does not rely on the research ledger, a search transcript, a floating-point optimizer, or an earlier author theorem.

Remark 1.4. The reserves $1/16$ and $1/40$ established below refer to constructed probability measures. They are not uniform lower bounds for ordinary uncovered density. Indeed, the classes $0 \pmod p$ for odd primes $p \le x$ have uncovered density $\prod_{3\le p\le x}(1 - 1/p)$, which tends to zero. For each fixed noncovering finite family, periodicity still supplies a whole uncovered residue class modulo its least common multiple.

2. The finite product and normalized distortion

Fix a finite family $\mathcal{F}$ of distinct odd moduli greater than one. Put

$$L = \operatorname{lcm}_{1\le i\le s} m_i = \prod_{p\in\mathcal{P}} p^{H_p}, \qquad \Omega_p = \mathbb{Z}/p^{H_p}\mathbb{Z}.$$

The empty family is immediate, so we may assume $s \ge 1$. The Chinese remainder theorem identifies $\mathbb{Z}/L\mathbb{Z}$ with $\prod_{p\in\mathcal{P}} \Omega_p$. A congruence modulo p^a, with $1 \le a \le H_p$, is called a *prefix of depth a* in the p-coordinate. Each class in $\mathcal{F}$ is a product of such prefixes and unrestricted coordinates.

Call a class *pure* if its modulus is a prime power, and *mixed* otherwise. Distinctness has two consequences that we will use. At any prime there is at most one pure class of each positive depth. At a fixed largest prime q, fixed old cofactor $d > 1$, and fixed current depth a, there is at most one class of modulus dq^a. These are statements about complete moduli, including all exponents.

Let λ_p be uniform measure on Ω_p, and let U_p be λ_p conditioned outside all the actual pure p-power classes.

Lemma 2.1. *The measure U_p exists. Every depth-a prefix A satisfies*

$$U_p(A) \le \frac{p-1}{(p-2)p^a}. \tag{2.1}$$

Proof. The removed λ_p-mass is at most $\sum_{a=1}^{H_p} p^{-a} < 1/(p-1)$. Its complement therefore has positive mass at least $(p-2)/(p-1)$. A depth-a prefix has unconditioned mass p^{-a}, which gives (2.1). □

Order the primes in $\mathcal{P}$ increasingly. Above an earlier point x, let $B_p(x) \subseteq \Omega_p$ be the union of the current prefixes of all mixed classes whose largest prime is p and whose earlier coordinates are satisfied by x. Write

$$\alpha_p(x) = U_p(B_p(x)).$$

For a fixed parameter $0 \le \delta_p < 1$, define the kernel density relative to U_p by

$$k_p(x,y) = \begin{cases} \dfrac{1}{1-\min\{\alpha_p(x),\delta_p\}}, & y \notin B_p(x), \\ \dfrac{(\alpha_p(x)-\delta_p)_+}{\alpha_p(x)(1-\delta_p)}, & y \in B_p(x). \end{cases} \tag{2.2}$$

The second value is set to zero when $\alpha_p(x) = 0$; that case concerns a U_p-null set. Starting with mass one on the empty product, let $\mu_{\le p}$ be the law obtained by successively applying these kernels. Write μ for the final law on $\mathbb{Z}/L\mathbb{Z}$. We write $X_{<p} = (X_r)_{r\in\mathcal{P},\ r<p}$ for the vector of earlier coordinates.

Lemma 2.2. *Every kernel in* (2.2) *has integral one and density at most $(1-\delta_p)^{-1}$. The joint marginal of all earlier coordinates is preserved. The probability of the mixed union ending at p is*

$$\int \frac{(\alpha_p(x)-\delta_p)_+}{1-\delta_p}\, d\mu_{<p}(x). \tag{2.3}$$

Put $C_p = (p-1)/((p-2)(1-\delta_p))$. For every prefix A of depth a,

$$\mathbb{P}_\mu(X_p \in A \mid X_{<p}) \le \min\left\{1, \frac{C_p}{p^a}\right\} \quad \textit{almost surely.} \tag{2.4}$$

Proof. If $\alpha \le \delta$, the bad set receives no mass and the good set has constant density $1/(1-\alpha)$. If $\alpha > \delta$, its two integrals are $(1-\alpha)/(1-\delta)$ and $(\alpha-\delta)/(1-\delta)$, whose sum is one. The claimed density bound follows in both cases. Integration at each fixed earlier point proves preservation of its marginal and (2.3). Combining the density bound with (2.1) gives (2.4) at the moment p is exposed. Later kernels preserve the entire joint law through p, and hence also preserve this conditional bound and the probability in (2.3). □

Every pure event has final probability zero. If b_p is an upper bound for (2.3), then the union bound gives

$$\mu\left(\bigcup \mathcal{F}\right) \le \sum_{p \in \mathcal{P}} b_p. \tag{2.5}$$

The law used here is the full normalized law. Conditioning it on survival is unnecessary and would not in general preserve the conditional caps.

3. A conditional convex comparison

We give the precise rearrangement statement needed for the dependent law constructed above. Independence will enter only in the auxiliary variables.

Lemma 3.1 (Nested marginals)**.** *Let $S \subseteq \{1, \dots, m\}$ be random, with $\mathbb{P}(j \in S) \le r_j$, where $1 \ge r_1 \ge \cdots \ge r_m \ge 0$. Let F be increasing and supermodular on subsets. Then*

$$\mathbb{E}F(S) \le \mathbb{E}F(\{j : U \le r_j\}), \tag{3.1}$$

where U is uniform on $[0,1]$.

Proof. Put $I_j = \{1, \dots, j\}$ and $I_0 = \varnothing$. The increasing-increment form of supermodularity, applied when the elements of S are inserted in this order, gives

$$F(S) \le F(\varnothing) + \sum_{j \in S} \big(F(I_j) - F(I_{j-1})\big).$$

All displayed increments are nonnegative. Taking expectations bounds the right side by $F(\varnothing) + \sum_j r_j(F(I_j) - F(I_{j-1}))$. This is the expectation of the nested set in (3.1). □

Proposition 3.2 (Conditional comparison)**.** *Let $X = (X_1, \dots, X_d)$ have any law on a finite product. For finitely many labels c, let $A_{c,i}$ depend only on coordinate i, and suppose*

$$\mathbb{P}(X_i \in A_{c,i} \mid X_1, \dots, X_{i-1}) \le r_{c,i} \quad \textit{almost surely,} \tag{3.2}$$

where the $r_{c,i} \in [0,1]$ are deterministic. If $w_c \geq 0$ and h is nonnegative, increasing and convex, then

$$\mathbb{E}h\left(\sum_c w_c \prod_i \mathbf{1}_{A_{c,i}}(X_i)\right) \leq \mathbb{E}h\left(\sum_c w_c \prod_i \mathbf{1}_{\{U_i \leq r_{c,i}\}}\right), \tag{3.3}$$

where the U_i are independent uniform variables on $[0,1]$.

Proof. Condition on $X_1, \ldots, X_{d-1}$. The integrand, as a function of the active labels in coordinate d, has the form $F(S) = h(\sum_{c \in S} v_c)$ with fixed $v_c \geq 0$. Convexity gives increasing increments, and monotonicity makes those increments nonnegative. Lemma 3.1 therefore replaces the last coordinate by a fresh independent uniform U_d, using (3.2).

For every fixed U_d, the new integrand is again h of a nonnegative rectangle load on the first $d-1$ coordinates. Their original marginal law still satisfies (3.2). Repeat at coordinate $d-1$ and continue backwards, then integrate all the independent uniforms. This proves (3.3). The argument concerns a finite label set. Subsequent enlargement to countable nonnegative loads is licensed by monotone convergence whenever the right side is finite. □

The proposition uses bounds conditional on the entire past. Bounds on unconditional marginals alone would not justify this backward induction. In the covering application, Lemma 2.2 supplies exactly (3.2); an unrestricted coordinate has cap one.

4. Complete cofactors and auxiliary heights

Fix the following rational threshold schedule at all odd primes. Unlisted primes, starting at 67, have threshold 1/2.

p	δ_p	p	δ_p	p	δ_p
3	0	19	11/17	43	31/41
5	0	23	16/25	47	34/45
7	1/5	29	19/27	53	19/25
11	1/3	31	18/25	59	37/50
13	6/11	37	37/50	61	37/50
17	29/50	41	29/39	$p \geq 67$	1/2

Table 1. The fixed physical thresholds. The last entry applies to primes.

For this schedule $C_p \leq p$ at every odd prime. At the listed primes this is a rational check; for $p \geq 67$ it follows from $C_p = 2(p-1)/(p-2) \leq p$. Let the K_p, indexed by all odd primes, be independent $\mathbb{N}_0$-valued auxiliary variables with

$$\mathbb{P}(K_p \geq a) = \frac{C_p}{p^a} \quad (a \geq 1). \tag{4.1}$$

Only finitely many of them occur in each expression. Equivalently,

$$\mathbb{P}(K_p = 0) = 1 - \frac{C_p}{p}, \qquad \mathbb{P}(K_p = k) = \frac{C_p(p-1)}{p^{k+1}} \quad (k \geq 1). \tag{4.2}$$

Realize these heights using independent uniforms V_p, so that $K_p \geq a$ if and only if $V_p \leq C_p/p^a$.

For $q \geq 5$ prime, put

$$Y_q = \sum_{3 \leq p < q} K_p, \qquad N_q = Y_q + \sum_{3 \leq p < r < q} K_p K_r, \qquad d_q = q - 2, \quad t_q = d_q \delta_q. \tag{4.3}$$

All sums and products over prime indices in the auxiliary calculations are over odd primes. The Rosser–Schoenfeld products in Section 8 explicitly include the prime 2.

There is no mixed charge at prime 3. At every larger ending prime we have the following bound.

Proposition 4.1 (Cofactor bound)**.** *If every class ending at q has at most three distinct prime divisors, then its mixed union has final probability at most*

$$b_q = \frac{\mathbb{E}(N_q - t_q)_+}{d_q - t_q}. \tag{4.4}$$

Proof. A mixed class ending at q has modulus dq^a with $d > 1$ using only smaller primes. Give this original class weight $w_{q,a} = (q-1)/q^a$, and let A_d denote its actual old-coordinate cylinder. The same d may occur at different current depths with different old residues; each such original class remains a separate label. By Lemma 2.1,

$$\alpha_q(x) \leq \frac{R_q(x)}{q-2}, \qquad R_q(x) = \sum_{dq^a \text{ in } \mathcal{F}} w_{q,a} \mathbf{1}_{A_d}(x),$$

where only mixed classes ending at q enter the sum. Monotonicity of the positive part bounds (2.3) by $\mathbb{E}(R_q - t_q)_+/(d_q - t_q)$.

Apply Proposition 3.2 with $h(u) = (u - t_q)_+$ and the prefix caps of Lemma 2.2. In the aligned load, a label with old cofactor $\prod_{p<q} p^{e_p}$ is active precisely when $e_p \leq K_p$ at all its nonzero coordinates. This depends on the complete cofactor exponents and no longer on its residue. Distinctness gives at most one original label per d and a, so for every fixed d its aligned total weight is at most

$$\sum_{a \geq 1} \frac{q-1}{q^a} = 1. \tag{4.5}$$

The allowed old cofactors have one or two distinct prime divisors. There are K_p of the form p^e and $K_p K_r$ of the form $p^e r^f$ in the completed auxiliary box. Their sum is N_q, proving (4.4).

If a smaller odd prime is absent from the actual family, inserting its independent auxiliary height only enlarges this nonnegative count. Completing

exponent ranges has the same effect. The actual law and the original label set remain finite. No mixed modulus can have largest prime 3, because every modulus is odd and its cofactor would have to contain a smaller odd prime. □

The bound retains original labels until the conditional comparison is complete. In particular, it does not assert distinctness of projected residue classes. Finite products of geometric heights have all the polynomial moments used below, so every expectation in (4.4) is finite.

Lemma 4.2. *For $j \geq 1$,*

$$\beta_j(p) := \mathbb{E}\binom{K_p}{j} = \frac{C_p}{(p-1)^j}, \qquad g_p := \mathbb{E}K_p = \frac{C_p}{p-1}. \tag{4.6}$$

Set $\beta_0(p) = 1$. On adjoining the next independent height K,

$$(N, Y) \longmapsto (N + (1+Y)K, Y + K). \tag{4.7}$$

The complete mean before q is

$$\mathbb{E}N_q = \sum_{p<q} g_p + \sum_{p<r<q} g_p g_r. \tag{4.8}$$

Proof. The tail-sum identity for binomial moments gives

$$\mathbb{E}\binom{K_p}{j} = \sum_{k \geq j} \binom{k-1}{j-1} \frac{C_p}{p^k} = \frac{C_p}{(p-1)^j}.$$

Adding K contributes K singleton types and YK two-prime types, giving (4.7). Independence yields (4.8). □

5. A positive finite moment recurrence

Throughout, $\binom{n}{j} = 0$ when $0 \leq n < j$, and a binomial coefficient with negative lower index is zero. For integers $r, s \geq 0$, define

$$B_{r,s}(z) = \binom{z}{r}\binom{r}{r+s-z} \quad (\max\{r,s\} \leq z \leq r+s), \tag{5.1}$$

and set it to zero otherwise. Counting two subsets by the size of their union gives

$$\binom{x}{r}\binom{x}{s} = \sum_z B_{r,s}(z)\binom{x}{z}. \tag{5.2}$$

For $i \geq 1$, let $G_i(u,v)$ be the number of i-element subsets of $[u] \times [v]$ meeting every row and column. It vanishes unless $1 \leq u, v \leq i$. Inclusion-exclusion gives the explicit integer formula

$$G_i(u,v) = \sum_{r=0}^{u}\sum_{s=0}^{v}(-1)^{r+s}\binom{u}{r}\binom{v}{s}\binom{(u-r)(v-s)}{i}. \tag{5.3}$$

For $i = 0$, set $G_0(0,0) = 1$ and all other values to zero. Selecting the rows and columns used by an i-element subset proves

$$\binom{(y+1)k}{i} = \sum_{u,v} G_i(u,v)\binom{y+1}{u}\binom{k}{v}. \tag{5.4}$$

Proposition 5.1 (Positive closure)**.** *The moments*

$$M_{a,b} = \mathbb{E}\left[\binom{N}{a}\binom{Y}{b}\right], \qquad a,b \geq 0, \quad a+b \leq 6,$$

form a closed system under (4.7)*. More precisely, after adjoining the independent height K_p,*

$$M'_{a,b} = \sum_{i=0}^{a}\sum_{\ell=0}^{b}\sum_{u,v} G_i(u,v) \sum_{\substack{r\in\{u,u-1\}\\ r\geq 0}} \sum_{z,h} B_{r,b-\ell}(z) B_{v,\ell}(h)\, M_{a-i,z}\, \beta_h(p). \tag{5.5}$$

Every coefficient is nonnegative, every old index satisfies $(a-i)+z \leq a+b$, and every height index satisfies $h \leq a+b$.

Proof. Apply Vandermonde's identity to each factor in

$$\binom{N+(Y+1)K_p}{a}\binom{Y+K_p}{b}.$$

The resulting terms are $\binom{N}{a-i}\binom{(Y+1)K_p}{i}\binom{Y}{b-\ell}\binom{K_p}{\ell}$. Use (5.4), Pascal's identity $\binom{Y+1}{u} = \binom{Y}{u} + \binom{Y}{u-1}$, and (5.2) in the Y and K_p variables. Independence then gives (5.5).

Nonnegativity follows from the subset interpretations. In every nonzero term, $r \leq u \leq i$, so $z \leq r+b-\ell \leq i+b-\ell$ and hence $(a-i)+z \leq a+b$. Likewise $h \leq v+\ell \leq i+\ell \leq a+b$. There are $\sum_{a=0}^{6}(7-a) = 28$ moment indices. The derivation is a polynomial identity, so it applies to all nonnegative heights without a truncation. □

Lemma 5.2 (Factorial bound for a positive part)**.** *For an integer $j \geq 2$ and a real $t \geq j-1$, let*

$$n_* = \left\lfloor \frac{jt}{j-1} \right\rfloor, \qquad \gamma_j(t) = \frac{n_* - t}{\binom{n_*}{j}}.$$

Then, for every integer $n \geq 0$,

$$(n-t)_+ \leq \gamma_j(t)\binom{n}{j}. \tag{5.6}$$

Proof. For $n < j$ the left side is zero. For $n \geq j$ and $n > t$, comparison of consecutive ratios shows that

$$\frac{n+1-t}{\binom{n+1}{j}} \geq \frac{n-t}{\binom{n}{j}} \quad\Longleftrightarrow\quad n+1 \leq \frac{jt}{j-1}.$$

Thus the maximum is attained at n_*; when $jt/(j-1)$ is an integer, the preceding index can tie. The assumptions imply $n_* \geq j$ and $n_* > t$, so the displayed coefficient is defined. □

For primes $71 \leq q \leq 89$ we use $j = 5$, and for $97 \leq q \leq 997$ we use $j = 6$. In these ranges $t_q = (q-2)/2 \geq j-1$. Therefore

$$b_q \leq \frac{\gamma_j(t_q) M_{j,0}^{(<q)}}{d_q - t_q}. \tag{5.7}$$

The choices are fixed mathematical inputs, rather than an optimization that must be trusted to succeed.

6. The exact finite certificate

6.1. **The initial charges at all heights.** The identity

$$\mathbb{E}(N-t)_+ = \mathbb{E}N - t + \mathbb{E}(t-N)_+ \tag{6.1}$$

reduces the exact calculation of a hinge to its complete mean and the states below t. All thresholds through prime 67 are at most $2183/50 < 50$. We retain the exact subdistribution

$$v_q(n,y) = \mathbb{P}(N_q = n, Y_q = y), \qquad 0 \leq y \leq n \leq 50.$$

Before prime 3 the only state is $(0,0)$ with mass one. On exposing p, each retained state sends mass

$$v(n,y)\mathbb{P}(K_p = k) \quad \text{to } (n+(y+1)k, y+k) \quad \left(0 \leq k \leq \left\lfloor \frac{50-n}{y+1} \right\rfloor\right). \tag{6.2}$$

States outside this region cannot return, since N is nondecreasing. The complete mean is maintained separately: if $A = \sum g_p$ and $E_2 = \sum_{p<r} g_p g_r$, their update is

$$(A, E_2) \longmapsto (A + g_p, E_2 + A g_p). \tag{6.3}$$

Consequently the exact initial charge at q is

$$b_q = \frac{A + E_2 - t_q + \sum_{n,y} v_q(n,y)(t_q - n)_+}{d_q - t_q}, \qquad 5 \leq q \leq 67. \tag{6.4}$$

Here A, E_2 have been accumulated only through primes smaller than q. Every omitted high state contributes through the complete mean. For example, this calculation gives

$$b_5 = \frac{1}{3}, \qquad b_7 = \frac{41}{180}, \qquad b_{11} = \frac{8930177}{83349000}.$$

6.2. **Directed rational rounding.** Put $\Lambda = 10^{24}$ and define

$$\operatorname{up}(x) = \frac{\lceil \Lambda x \rceil}{\Lambda}.$$

Start the 28 moments at $M_{0,0} = 1$ and all other entries zero, before prime 3. At every prime through 997, substitute the current rational upper moments in (5.5) and round each outgoing entry upward with up. Positivity of the recurrence proves inductively that every stored entry remains an upper bound for the complete moment.

For $q \le 67$, round the exact charge (6.4) upward. For $71 \le q \le 997$, use the incoming stored upper moment in (5.7), then round the charge upward. Denote the sum of these rational upper charges by H_{1000}. The cutoff includes all odd primes at most 1000; the last is 997.

Proposition 6.1 (Finite arithmetic certificate)**.** *The prescribed calculation has 167 odd primes, 166 mixed-charge entries, and 28 moment entries. It gives*

$$H_{1000} = \frac{936901507790779059055101}{10^{24}}, \tag{6.5}$$

$$\sum_{p \le 1000} g_p \le \frac{425099311283849479212119 4}{10^{24}} < \frac{4251}{1000}. \tag{6.6}$$

It also certifies

$$\prod_{3 \le p \le 1000} \left(1 + C_p \frac{3p-1}{(p-1)^2}\right) < 28851. \tag{6.7}$$

Exact-arithmetic verification. The verifier uses Table 1, the height law (4.2), the moments (4.6), and the positive recurrence of Proposition 5.1. It calculates the initial charges from (6.2)–(6.3), with rounding scale Λ. The transition coefficients are reconstructed as nonnegative integers and the primes are enumerated by a sieve. Table 2 lists the three groups of accumulated charge numerators. Their sum gives (6.5).

The $(0, 1)$ entry after the last prime gives (6.6). The product in (6.7) is evaluated as an exact fraction and compared with the integer 28851. The code, complete charge trace, and final moment vector are included in the source package. Appendix A gives the reconstruction instructions and a second implementation based on ordinary-power intervals. All comparisons in this proposition are integer or rational comparisons. □

Primes	Charge calculation	Numerator over 10^{24}
5–67	Exact low states	911526588327781710324674
71–89	Fifth factorial moment	18325457568679109084577
97–997	Sixth factorial moment	7049461894318239645850

TABLE 2. The finite certificate, grouped by the bound used.

7. Completion of the entire prime tail

The remainder requires no enumeration of primes and no height cutoff. Set $B = 1000$. We use the rational mean ceiling $4251/1000$. An alternative using only finite rational calculations is proved in Appendix B; its slightly larger tail bound gives the same upper bound (1.1) for the total charge.

Lemma 7.1 (A polynomial moment majorant)**.** *For $A_q = \sum_{p<q} g_p$,*

$$\mathbb{E}\binom{N_q}{6} \le P(A_q), \tag{7.1}$$

where the positive polynomial of degree 12 is

$$P(A) = \sum_{h=1}^{12} a_h \sum_{r=1}^{h} \frac{\binom{h-1}{r-1}}{r!\,2^{h-r}} A^r, \tag{7.2}$$

and its integer coefficients a_h are given by

$$\begin{aligned}(a_0, a_1, \dots, a_{12}) = (0, 0, 0, 1, 206, 3965, 27364, 94682,\\ 187628, 223650, 159075, 62370, 10395).\end{aligned} \tag{7.3}$$

Proof. Since $\sum K_p^2 \ge \sum K_p = Y_q$, we have

$$N_q = Y_q + \frac{Y_q^2 - \sum_{p<q} K_p^2}{2} \le \frac{Y_q(Y_q+1)}{2}. \tag{7.4}$$

There are $y(y+1)/2$ singleton or unordered-pair subsets of a y-set. Count six distinct such subsets by the size h of their union. If a_h counts the selections using every point of a fixed h-set, then

$$\binom{y(y+1)/2}{6} = \sum_{h=0}^{12} a_h \binom{y}{h}. \tag{7.5}$$

These counts are nonnegative and vanish above 12. Inclusion-exclusion computes them explicitly as

$$a_h = \sum_{u=0}^{h} (-1)^{h-u} \binom{h}{u} \binom{u(u+1)/2}{6}, \tag{7.6}$$

giving (7.3).

Write $\preceq$ for coefficientwise inequality of formal power series. The binomial moments (4.6) and independence give

$$\begin{aligned}\mathbb{E}(1+z)^{Y_q} &= \prod_{p<q} \left(1 + \frac{g_p z}{1 - z/(p-1)}\right)\\ &\preceq \exp\left(\frac{A_q z}{1 - z/2}\right).\end{aligned} \tag{7.7}$$

Indeed, $1 + f \preceq \exp(f)$ for a series f with nonnegative coefficients and zero constant term, and $p \geq 3$. Combine (7.4)–(7.5) with the coefficient identity

$$[z^h] \exp\left(\frac{A_q z}{1 - z/2}\right) = \sum_{r=1}^{h} \frac{\binom{h-1}{r-1}}{r!\, 2^{h-r}} A_q^r \quad (h \geq 1). \qquad \square$$

Lemma 7.2 (All-prime tail bound)**.** *The charges in* (4.4) *satisfy*

$$\sum_{q>1000} b_q \leq T := \frac{250000}{81 \cdot 988^5} \sum_{h=0}^{12} \frac{2^h P^{(h)}(4251/1000)}{5^{h+1}} < \frac{174}{10^6}. \tag{7.8}$$

Proof. For $q > 1000$ we have $t_q = (q-2)/2$. Since $\binom{n}{6} \geq (n-5)^6/720$ for $n \geq 6$, maximizing $(z-a)/z^6$ over $z > a > 0$ gives

$$\frac{\gamma_6(t_q)}{d_q - t_q} \leq \frac{250000}{81(q-2)(q-12)^5} \leq \frac{250000}{81(q-12)^6}. \tag{7.9}$$

For the first inequality take $a = t_q - 5 = (q-12)/2$; the maximum is $5^5/(6^6 a^5)$. Lemmas 5.2 and 7.1 now bound b_q by the last coefficient times $P(A_q)$.

For primes above B, $g_p = 2/(p-2)$. Enlarging the sum to integers and integrating the decreasing reciprocal function gives

$$\begin{aligned} A_q &\leq \frac{4251}{1000} + 2\log\frac{q-2}{B-2} \\ &\leq \frac{4251}{1000} + 2\log\frac{q-12}{B-12}. \end{aligned} \tag{7.10}$$

For example, $\sum_{B<n<q}(n-2)^{-1} \leq \int_B^q (x-2)^{-1}\,dx$ proves the first step, using (6.6). The second follows by cross-multiplication of the positive ratios when $q \geq B$.

The polynomial P has nonnegative coefficients and degree at most 12, so $AP'(A) \leq 12P(A)$ for $A > 0$. For $A \geq 4251/1000 > 4$ this gives $2P'(A) \leq 6P(A)$. Hence

$$f(x) = \frac{P(4251/1000 + 2\log((x-12)/(B-12)))}{(x-12)^6}$$

is decreasing on $[B, \infty)$. We may therefore enlarge the prime sum to all integers greater than B and then integrate:

$$\sum_{q>B} b_q \leq \frac{250000}{81} \int_B^\infty f(x)\,dx.$$

Substitute $u = \log((x-12)/(B-12))$ and expand the polynomial exactly. Using $\int_0^\infty u^h e^{-5u}\,du = h!/5^{h+1}$ gives the expression T in (7.8). Its exact rational value is

$$T = \frac{16728968874809530693336175552877798571687006569}{964016971193516032000000000000000000000000000000} < \frac{174}{10^6}. \tag{7.11}$$

The inequality is an integer comparison, independently reconstructed by both accompanying implementations. □

Proof of Theorem 1.1. Apply the normalized kernels with Table 1 to the finite actual family. Each mixed ending event has probability at most b_q by Proposition 4.1, and later kernels preserve this probability. Primes absent from the family contribute only nonnegative auxiliary upper terms. Propositions 5.1 and 6.1, followed by Lemma 7.2, give

$$\mu\left(\bigcup \mathcal{F}\right) \le H_{1000} + T < \frac{937076}{10^6} < \frac{15}{16}.$$

Pure classes have probability zero. Thus the finite CRT product contains an uncovered point, which corresponds to an uncovered integer. In fact, the constructed law gives the uncovered set mass greater than 1/16. □

8. Unrestricted moduli after a fixed largest prime

The same comparison also gives a direct continuation for moduli with arbitrarily many prime divisors. Only the cofactor count changes. For any real $x \ge 3$, put

$$D_q = \prod_{3 \le p < q} (1 + K_p), \qquad S(x) = \prod_{3 \le p \le x} \left(1 + C_p \frac{3p-1}{(p-1)^2}\right). \tag{8.1}$$

These are auxiliary quantities belonging to the same fixed schedule; $S(x)$ is deterministic. The physical coordinate heights H_p still resolve the entire original family, including all classes ending after the cutoff.

Lemma 8.1 (Unrestricted second-moment charge)**.** *At any prime $q \ge 67$, without a restriction on the number of prime divisors of the moduli ending there, their mixed union has probability at most*

$$\frac{\mathbb{E} D_q^2}{(q-2)^2}. \tag{8.2}$$

Furthermore,

$$\mathbb{E}(1 + K_p)^2 = 1 + C_p \frac{3p-1}{(p-1)^2} = \frac{p(p+3)}{(p-1)(p-2)} \le \left(\frac{p}{p-1}\right)^6 \quad (p \ge 67). \tag{8.3}$$

Proof. Apply Proposition 3.2 to the original labels, and complete all nonempty cofactor types as in the proof of Proposition 4.1. There are $D_q - 1$ types in the completed box. Each has total weight at most one. Thus the mixed probability is bounded by

$$\frac{\mathbb{E}(D_q - 1 - t_q)_+}{d_q - t_q}.$$

For $t > 0$ and $u \ge 0$, $(u-t)_+ \le u^2/(4t)$. With $t_q = d_q/2$, the last expression is at most $\mathbb{E}(D_q - 1)^2/d_q^2 \le \mathbb{E} D_q^2/d_q^2$, proving (8.2).

By (4.6), $\mathbb{E}(1+K_p)^2 = 1 + 3\mathbb{E}K_p + 2\mathbb{E}\binom{K_p}{2}$, which gives the first equality in (8.3). The second uses $\delta_p = 1/2$. For the inequality, clear positive denominators. With $x = p-3 \geq 0$, the resulting difference is

$$p^5(p-2) - (p+3)(p-1)^5 = 51 + 136x + 115x^2 + 40x^3 + 5x^4 > 0.$$

Independence multiplies these factors to evaluate $\mathbb{E}D_q^2$. □

We use the following explicit classical prime-product estimate solely for the unrestricted continuation. Rosser and Schoenfeld [13, Theorem 8, equations (3.28)–(3.29)] prove

$$e^\gamma \log x \left(1 - \frac{1}{2\log^2 x}\right) < \prod_{p \leq x} \frac{p}{p-1} \qquad (x > 1),$$

$$\prod_{p \leq x} \frac{p}{p-1} < e^\gamma \log x \left(1 + \frac{1}{2\log^2 x}\right) \qquad (x \geq 286).$$

Here the products include the prime 2. Taking a ratio cancels its factor, as well as e^γ.

Lemma 8.2 (Explicit prime-product ratio)**.** *Let $B \geq 286$ and let $\ell \geq 3$ be an integer with $3^\ell \leq B$. Then, for $T \geq B$,*

$$\prod_{B < p \leq T} \frac{p}{p-1} \leq c_\ell \frac{\log T}{\log B}, \qquad c_\ell = \frac{2\ell^2+1}{2\ell^2-1}. \tag{8.4}$$

Proof. Since $e < 3$, the hypothesis gives $\ell < \log B$. In the ratio of the displayed Rosser–Schoenfeld inequalities, the numerator correction is at most $1 + 1/(2\log^2 B)$ and the denominator correction is $1 - 1/(2\log^2 B) > 0$. Replacing $\log B$ by its lower bound ℓ in this correction ratio yields c_ℓ. □

Proposition 8.3 (Complete unrestricted tail)**.** *Under the hypotheses of Lemma 8.2, with B an integer, the sum of the mixed ending-event probabilities at primes $q > B$, with arbitrary moduli at those primes, is at most $S(B)\tau(B,\ell)$, where*

$$\tau(B,\ell) = \frac{c_\ell^6}{B}\left(\frac{B}{B-3}\right)^2 \sum_{h=0}^{6} \frac{6!}{(6-h)!\,\ell^h}. \tag{8.5}$$

Proof. Equations (8.1)–(8.3) and Lemma 8.2 give

$$\mathbb{E}D_q^2 \leq S(B) c_\ell^6 \left(\frac{\log q}{\log B}\right)^6.$$

It is harmless that the prime product used here includes q, whereas D_q includes only primes smaller than q. Also,

$$\frac{1}{(q-2)^2} \leq \frac{1}{(q-3)^2} \leq \left(\frac{B}{B-3}\right)^2 \frac{1}{q^2}.$$

The function $(\log x)^6/x^2$ is decreasing for $\log x \geq 3$. Enlarge the prime sum to integers and integrate from B to obtain

$$\sum_{q>B} \frac{(\log q/\log B)^6}{q^2} \leq \int_B^\infty \frac{(\log x/\log B)^6}{x^2}\,dx = \frac{1}{B}\sum_{h=0}^{6} \frac{6!}{(6-h)!\,(\log B)^h}.$$

The equality follows from $u = \log(x/B)$ and $\int_0^\infty u^h e^{-u}\,du = h!$. Replace $\log B$ by the lower bound ℓ in the positive sum and apply Lemma 8.1. All estimates are for the same full law and remain valid if only a finite subfamily of later primes is present. □

Proof of Theorem 1.2. Set $B = 10^9$ and apply the fixed kernels to the entire given family. For every ending prime at most B, the three-divisor hypothesis holds. Thus the sum of all those event probabilities is less than $\Gamma_0 = 937076/10^6$: their partial auxiliary charge sum is bounded by the full three-divisor series already estimated above.

To estimate $S(B)$, use (6.7) and the ratio Lemma 8.2 from 1000 to 10^9, with $\ell = 6$. Here $3^6 < 1000$ and $\log(10^9)/\log(1000) = 3$. Hence

$$S(B) < K_B := 28851\left(\frac{73}{71}\right)^6 3^6. \tag{8.6}$$

For the tail after B, take $\ell = 18$, since $3^{18} < 10^9$. The entirely rational comparison

$$K_B\tau(10^9, 18) < \frac{36711}{10^6} \tag{8.7}$$

is verified by both implementations. It follows that the final uncovered mass is greater than

$$1 - \frac{937076}{10^6} - \frac{36711}{10^6} = \frac{26213}{10^6} > \frac{1}{40}.$$

Pure classes have probability zero as before. The finite CRT product therefore contains an uncovered point. No primes between 1000 and 10^9, or beyond 10^9, have been enumerated in this argument. □

Appendix A. Exact reconstruction and arithmetic cross-check

The accompanying source package contains `verify.py`, `crosscheck.py`, and `certificate.json`. Both programs require only the Python standard library and use rational numbers and integers for every decisive comparison. Run, from the package directory,

```
python3 verify.py
python3 crosscheck.py
```

The verifier recomputes the certificate from the printed inputs and then compares it with the stored file. The stored values are not proof premises. To regenerate the file in a fresh source directory, use

```
python3 verify.py --write certificate.json
```

The fixed prime list ends at 997. The positive transition coefficients are reconstructed from (5.1) and (5.3); there is no precomputed transition table or external solver dependency. The certificate records all 166 rounded charge entries and the final 28-component moment vector, in units of 10^{-24}. The finite numerical facts needed by the proof are Propositions 6.1, (7.11), and (8.7).

A.1. **The second implementation.** The cross-check neither imports the verifier nor reads its certificate. It enumerates primes by trial division and represents the initial low states by (Y, Z), where $Z = \sum K_p^2$, recovering $N = Y + (Y^2 - Z)/2$. Its low-state update is

$$(Y, Z) \longmapsto (Y + K, Z + K^2).$$

It uses the complete means, as in (6.1), so no omitted high state is discarded from the hinge expectation.

For the moment continuation it propagates lower and upper endpoints for the ordinary-power moments $R_{a,b} = \mathbb{E}[N^a Y^b]$, $a + b \le 6$, in units of 10^{-32}. Direct expansion of (4.7) yields

$$R'_{a,b} = \sum_{i=0}^{a} \sum_{\ell=0}^{b} \sum_{r=0}^{i} \binom{a}{i} \binom{b}{\ell} \binom{i}{r} R_{a-i,b-\ell+r}\, \mathbb{E}K_p^{i+\ell}. \tag{A.1}$$

All coefficients are nonnegative and all indices remain within total order six. The height inputs are

$$\mathbb{E}K_p^j = \sum_{h=1}^{j} \left\{ {j \atop h} \right\} h!\, \frac{C_p}{(p-1)^h} \quad (j \ge 1),$$

with zeroth moment one. Braces denote Stirling numbers of the second kind. The lower endpoints are rounded downward and the upper endpoints upward.

To bound $\mathbb{E}\binom{N}{j}$, expand $\binom{n}{j}$ in ordinary powers. A positive coefficient multiplies an upper endpoint and a negative coefficient a lower endpoint. This sign rule is essential for a rigorous upper bound. The resulting finite charge differs from (6.5) by less than 10^{-20} and independently certifies the same strict inequality (1.1).

For the triple tail, the cross-check obtains the a_h by univariate Newton differences and integrates the direct expansion of $P(4251/1000 + 2u)$ against e^{-5u}. It also recomputes the head second-moment product and the unrestricted rational continuation. Thus it checks the arithmetic through a different moment basis and different state representation. The proofs of conditional comparison, cofactor completion, positive closure, and the infinite-tail estimates are supplied in the body of the paper; numerical agreement alone is not used to justify those reductions.

Appendix B. A finite rational tail and its formal verification

This appendix gives an alternative proof of Theorem 1.1 in which every probability space is finite and the tail is bounded by a telescoping sequence of rational numbers. It uses the same threshold schedule and the same polynomial P as the main text. The resulting certificate is the one checked in Lean. The integral proof in Section 7 remains valid and gives a slightly smaller bound.

B.1. **Finite heights and complete labels.** For the given finite family choose an integer $D \geq 51$ at least as large as every prime exponent in its moduli. Replace each auxiliary height by $K_p^{[D]}$ supported on $\{0, \dots, D\}$, with law

$$\begin{aligned} \mathbb{P}(K_p^{[D]} = 0) &= 1 - C_p/p, \\ \mathbb{P}(K_p^{[D]} = k) &= C_p(p-1)/p^{k+1} \quad (1 \leq k < D), \\ \mathbb{P}(K_p^{[D]} = D) &= C_p/p^D. \end{aligned} \tag{B.1}$$

Admissibility follows from $0 \leq C_p \leq p$. Its survival probability is C_p/p^a for $1 \leq a \leq D$, so all prefix bounds needed by the actual family are unchanged. For every $j \geq 1$, the finite tail-sum identity gives

$$\mathbb{E}\binom{K_p^{[D]}}{j} = C_p \sum_{k=j}^{D} \binom{k-1}{j-1} p^{-k} \leq \frac{C_p}{(p-1)^j}. \tag{B.2}$$

For completeness, let $S_j(D)$ denote the sum in this display without the factor C_p. The finite geometric-sum formula and Pascal's identity give

$$S_1(D) = \frac{1 - p^{-D}}{p-1}, \qquad (p-1)S_{j+1}(D) = S_j(D) - \binom{D}{j} p^{-D}.$$

Induction proves $S_j(D) \leq (p-1)^{-j}$ for every D, including $j > D$. Thus the positive moment recurrence continues to give upper bounds. The finite sum of current depth weights is at most one, so Proposition 4.1 also applies to these finite heights.

For the head, insert any absent odd primes through 997. For the continuation, insert comparison indices $1000, 1001, 1002, \dots$ up to any finite endpoint containing the family. At an inserted index extend every actual exponent vector by zero and every prefix condition by the always-true condition. This preserves the support size and injectivity of complete exponent vectors. It also preserves coverage: restricting a point to the actual coordinates finds an original class, whose added conditions are automatic. These extra indices introduce no arithmetic congruences or additional CRT requirements. Their independent auxiliary heights only enlarge the nonnegative completed cofactor count. For every integer index $q \geq 1000$, use $\delta_q = 1/2$, $C_q = 2(q-1)/(q-2)$, and $g_q = 2/(q-2)$. The finite law (B.1) is admissible at these indices as well.

The actual distortion kernels still condition on the entire preceding history. Original class labels are retained until the backward comparison is complete; injectivity is never asserted for projected residue classes. The choice of D depends on the given family and imposes no uniform exponent bound.

B.2. **A finite polynomial moment envelope.** Set $B_0(A) = 1$ and, for $1 \le h \le 12$, define

$$B_h(A) = \sum_{r=1}^{h} \frac{\binom{h-1}{r-1}}{r!\,2^{h-r}} A^r. \tag{B.3}$$

Thus $P(A) = \sum_{h=0}^{12} a_h B_h(A)$, with the coefficients (7.3).

Lemma B.1. *Let $K_1, \dots, K_m$ be independent nonnegative integer-valued random variables on finite probability spaces. Suppose $g_i \ge 0$ and*

$$\mathbb{E}\binom{K_i}{j} \le g_i/2^{j-1} \qquad (1 \le j \le 12).$$

For $Y = \sum_i K_i$ and $A = \sum_i g_i$,

$$\mathbb{E}\binom{Y}{h} \le B_h(A) \qquad (0 \le h \le 12).$$

Consequently, for $N = \sum_i K_i + \sum_{i<j} K_i K_j$, $\mathbb{E}\binom{N}{6} \le P(A)$.

Proof. For $h \ge 1$, comparison of coefficients in (B.3) gives the derivative identity below. The constant coefficients on both sides are 2^{1-h}; the others follow from $\sum_{v=r-1}^{h-2} \binom{v}{r-1} = \binom{h-1}{r}$ for $1 \le r \le h-1$:

$$B_h'(A) = \sum_{j=1}^{h} 2^{1-j} B_{h-j}(A) \qquad (h \ge 1).$$

All coefficients of B_h are nonnegative. Finite Taylor expansion therefore gives, for $A, g \ge 0$,

$$B_h(A) + \sum_{j=1}^{h} \frac{g}{2^{j-1}} B_{h-j}(A) = B_h(A) + gB_h'(A) \le B_h(A+g). \tag{B.4}$$

The assertion is immediate for no variables. On adjoining K_i, Vandermonde's identity and independence express the new hth binomial moment as the convolution of the old moments with those of K_i. Apply the inductive bounds and (B.4); the zeroth moment remains one. Finally use (7.4), monotonicity of $\binom{n}{6}$ on nonnegative integers, and the nonnegative expansion (7.5). □

For the height at comparison index q, (B.2) gives the required input with $g_q = C_q/(q-1)$, since every comparison index is at least three. This proves the same polynomial majorant as Lemma 7.1 using only finite sums and polynomial identities.

B.3. **A rational telescoping potential.** Define the degree-twelve polynomial

$$\mathcal{H}(A) = \sum_{r=0}^{12} \frac{2^r P^{(r)}(A)}{5^{r+1}}. \tag{B.5}$$

It has nonnegative rational coefficients and satisfies the polynomial identity

$$5\mathcal{H} - 2\mathcal{H}' = P. \tag{B.6}$$

The derivatives here are formal derivatives of polynomials.

Lemma B.2 (Potential inequality)**.** *For rational $A \geq 0$ and $z > 0$,*

$$\frac{P(A)}{(z+1)^6} \leq \frac{\mathcal{H}(A)}{z^5} - \frac{\mathcal{H}(A + 2/(z+11))}{(z+1)^5}. \tag{B.7}$$

Proof. Every derivative of P and $\mathcal{H}$ is nonnegative at $A \geq 0$. Differentiate (B.6) to obtain $2\mathcal{H}^{(r+1)}(A) \leq 5\mathcal{H}^{(r)}(A)$. In particular, $\mathcal{H}^{(r)}(A) \leq (5/2)^{r-1}\mathcal{H}'(A)$ for $r \geq 1$. Put $s = 2/(z+11)$ and $\eta = 5/(z+11)$, so $0 < \eta < 1$. Finite Taylor expansion, $r! \geq 1$, and the finite geometric-sum formula give

$$\begin{aligned} \mathcal{H}(A+s) &\leq \mathcal{H}(A) + s\mathcal{H}'(A) \sum_{r=1}^{12} \frac{\eta^{r-1}}{r!} \\ &\leq \mathcal{H}(A) + \frac{s}{1-\eta}\mathcal{H}'(A) = \mathcal{H}(A) + \frac{2}{z+6}\mathcal{H}'(A). \end{aligned} \tag{B.8}$$

Here $(1-\eta)\sum_{r=0}^{11} \eta^r = 1 - \eta^{12} \leq 1$; no infinite series is needed. Using (B.6) and (B.8), the numerator obtained by clearing the positive denominator $z^5(z+1)^6$ in (B.7) satisfies

$$\begin{aligned} &(z+1)^6\mathcal{H}(A) - (z+1)z^5\mathcal{H}(A+s) - z^5 P(A) \\ &\quad \geq (15z^4 + 20z^3 + 15z^2 + 6z + 1)\mathcal{H}(A) + \frac{10z^5}{z+6}\mathcal{H}'(A) \geq 0. \end{aligned}$$

This proves the inequality. □

Set $A_0 = 4251/1000$ and, for $n \geq 0$, put

$$\begin{aligned} A_{n+1} &= A_n + \frac{2}{998+n}, \\ T_n &= \frac{250000}{81}\,\frac{\mathcal{H}(A_n)}{(987+n)^5}, \qquad Q_n = \frac{250000}{81}\,\frac{P(A_n)}{(988+n)^6}. \end{aligned} \tag{B.9}$$

The sum of the height-mean bounds before index $1000+n$ is at most A_n: the head is bounded by (6.6), and each subsequent index contributes $2/(q-2)$. The scalar calculation in (7.9) holds for every $q > 12$, whether or not q is prime. Lemma B.1 and that sixth-hinge bound therefore bound the auxiliary charge at this index by Q_n.

Proposition B.3 (Finite rational tail)**.** *Every finite continuation has total auxiliary charge at most* $\widehat{T} := T_0$*, where*

$$\widehat{T} = \frac{26553918848904016973549485004567934240773026 3}{15224574739239936000000000000000000000000000000000}. \tag{B.10}$$

Moreover, $\widehat{T} < 175/10^6$*.*

Proof. Apply Lemma B.2 with $A = A_n$ and $z = 987 + n$. It gives $Q_n \le T_n - T_{n+1}$. Since $T_n \ge 0$, for every $M \ge 0$,

$$\sum_{n=0}^{M-1} Q_n \le T_0 - T_M \le T_0.$$

Substitution in (B.5) gives the exact fraction (B.10); its upper bound is an integer comparison. The endpoint M is arbitrary and may be chosen to include every prime in the original family. □

In terms of the integral bound T of (7.8),

$$\widehat{T} = \left(\frac{988}{987}\right)^5 T. \tag{B.11}$$

The integer comparison schedule starts at 1000, including that unused composite index. Its initial potential denominator is consequently 987^5. In particular, (B.10) does not use the sharper intermediate bound $174/10^6$ from Lemma 7.2.

B.4. **The formal certificate and its numerical comparison.** For the early charges, the formal certificate rounds every retained low-state mass upward to a multiple of 10^{-24} after each transition (6.2). It uses the incoming upper moment $\widehat{M}_{1,0}$ for the complete mean in (6.1), and rounds the resulting charge upward to a multiple of 10^{-18}. All transition coefficients and the complementary-hinge weights are nonnegative, so both replacements preserve upper bounds. For $D \ge 51$ the atoms below 51 in (B.1) equal those in (4.2). Also $Y \le N$ and every early threshold is less than 50. Thus omitted states have zero complementary hinge; their contribution to the original hinge remains in the complete mean.

The early charge ceilings sum to $911526588327781719/10^{18}$. The later charges, for $71 \le q \le 997$, are the same ceilings as in Section 6. The resulting head bound is

$$\widehat{H}_{1000} = \frac{936901507790779067730427}{10^{24}} = H_{1000} + \frac{8675326}{10^{24}}. \tag{B.12}$$

The sum, compared directly as a rational number, satisfies

$$\widehat{H}_{1000} + \widehat{T} < \frac{937076}{10^6} < \frac{15}{16}. \tag{B.13}$$

The separate bound $\widehat{T} < 175/10^6$ is not substituted to obtain (B.13); the exact fraction is used.

Certificate	Tail, approximately	Total, approximately
Integral argument	0.0001735339664622	0.9370750417572413
Formal rational argument	0.0001744148477295	0.9370759226385085

TABLE 3. Two exact certificates for the same strict total bound. The decimal values are explanatory; all decisive comparisons use the exact fractions.

Combining the finite-height comparison with (B.13) gives an alternative proof of Theorem 1.1. Under the constructed normalized law, the union of all mixed classes has mass less than $15/16$, and pure classes have mass zero. Restrict an uncovered point to the actual prime coordinates and apply the finite CRT. This yields an uncovered integer, with no restriction on the family size, its total prime support, or its exponents.

B.5. **Formal statement and reproducibility.** The public Lean declaration in namespace `Erdos7` is

`noncoverage_at_most_three_prime_factors`.

Its arguments are a finite family of integer residues and natural moduli, with exactly the assumptions in Theorem 1.1; its conclusion is the existence of an integer outside every class. The proof constructs the probability laws and proves the comparisons and certificates. No residual-model, certificate-validity, or height-bound hypothesis is supplied to the final theorem.

The project uses Lean 4 [5] and mathlib [10], with both pinned to `v4.32.0-rc1`; the dependency manifest records the exact mathlib commit. The final axiom audit reports exactly `propext`, `Classical.choice`, and `Quot.sound`. There are no admitted proofs, additional mathematical axioms, or uses of native evaluation as a proof oracle. For the potential inequality, Lean checks the same statement as Lemma B.2 by clearing denominators and verifying a polynomial expansion with 221 nonnegative coefficients. The finite head inequalities are checked by kernel reduction. The supplied Python generators propose source files; they are not trusted proof premises.

The companion includes a build script and an audit of the arithmetic statements and their axiom dependencies. With the pinned toolchain and dependencies installed, run from its `formal/` directory:

```
python3 build_three_prime.py --jobs 4
lake env lean --trust=0 AuditThreePrime.lean
```

The supplementary `verify_formal_appendix.py`, run beside `verify.py`, reconstructs the rounded formal head, the polynomial identities, and the exact comparison in Table 3. It is an additional arithmetic check, not an independent implementation of the entire proof. Theorem 1.2 and Corollary 1.3, with their 10^9 cutoff, are not part of the completed Lean formalization.

Acknowledgments

This work benefited from research assistance by OpenAI's AI systems, including proof exploration, exact computational checks, and manuscript preparation.

Availability and responsibility

The source package contains the complete manuscript, two standalone arithmetic implementations, the reconstructed certificate, and build and verification instructions. No earlier author paper is required to verify either theorem.

The manuscript and matching source package are available from Zenodo under DOI 10.5281/zenodo.22760638.

The companion Lean 4 source proves Theorem 1.1 end to end; Appendix B gives the rational alternative and the precise verification scope. Theorem 1.2 and its 10^9 cutoff are not included in that formal verification. The author is responsible for the mathematical statements and their presentation.

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