TECHNICAL SUPPLEMENT
NEURAL COMPUTATION:
OPTIMAL PAIRWISE-MAX RELU REPRESENTATIONS

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Abstract. This supplement supplies the detailed arguments for the exact directional widths $W^*(5,2) = 6$ and $W^*(6,2) = 33$ in the pairwise-max ReLU model. It develops the finite five-input classification, the six-input canonical transfer, the local support laws, and the complete lower-bound case analysis. For each finite computation, the evidence register states the predicate, the family it covers, and the exact-arithmetic check. Both rational optimality results have complete Lean proofs. Section S12 records their scope and verification, including the MAX6 source rebuild of all 16,057 PM2 dependency modules and independent kernel replay of the closed theorem and its all-real projections. The extension to real coefficients follows from the separate rationalization lemma in the main paper.

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S1. Reading guide and proof conventions

The main paper gives the model, the all-real chamber criterion, both complete constructions, the canonical transfer, and the global lower-bound argument. This supplement supplies the finite classifications and local case proofs on which that argument depends. The completed Lean development establishes rational PM2 optimality from the original dictionary through the final case assembly. Throughout, the model uses the fixed pairwise-max dictionary and a free feature-linear term.

By the rational-feasibility lemma in the main paper, it suffices to prove the lower bounds for rational coefficients. Throughout the proof those coefficients may have either sign. A support is a set of actual nonzero projective columns. In particular, canceling N entries still cost support until the representation itself has been transformed. Mathematical coordinates are numbered $1, \dots, 6$. Literal registry IDs and row keys retain the archive's zero-based labels; every use is marked. The factor-three scaling is used from Section S4 onward.

Reader's purpose	Route through the supplement
Check MAX5	S2, then the formal verification record in S12.
Check the reduction	S3 and the raw-model requirements in S11.
Understand MAX6	S4–S5 for notation and local costs; S6–S10 for exclusions.
Audit finite evidence	S11's predicates and source register; S12's execution records.
Check formal scope	S12.3 and S12.5 for the theorem types; the correspondence map in S12.7 for source declarations.
Reproduce the identities	S12.1 and the implementation register in S12.8.

What a finite lemma means. A finite lemma has two obligations. Its input family must contain every object allowed by the analytic hypotheses, and its arithmetic must establish the claimed predicate on every member. A row separator for an undersized column set proves nothing about omitted columns. Likewise, a list of cost tuples routes cases but does not classify rational representations. The numbered source register in S11 makes the active source, its scope, and its finite records separately identifiable.

Order of proof. Read the dependencies in the following order. Here e is the correction support and R the retained support, so $W = e + R$.

Step	Inputs and resulting bound
Local foundation	Canonical transfer (S3), local laws (S4–S5), and the retained bound $R \geq 22$ (S6.1).
Preliminary chain	S13 proves $W \geq 27, 28, 29, 30, 31$ in that order. Each strict support decrease uses only an earlier bound or a chosen minimum.
Independent branches	S7 proves the nonmatching exclusions under $R \leq 32 - e$, without assuming $W \geq 32$; S14 supplies the matching retained-28 cases.
H32	S6 combines those branches with $W \geq 31$ to exclude width 31.
Final exclusion	After H32, S8–S10 and S15 exclude matching $R = 29$; S7 and the correction census exclude the other width-32 cases.

In particular, H32 does not use the later matching retained-29 conclusion. Lexicographic minimization at fixed correction data uses integer support counts; a strict decrease is justified only within that minimum or by an already established bound.

Detailed derivations. Sections S13–S15 contain the full component calculations behind the earlier lower bounds, the retained-28 helper cover, and the remaining retained-29 cases. The first ten sections provide the conceptual route and reusable local lemmas. Sections S11–S12 specify the finite predicates and their execution. Numbered lemmas have the global prefix SL; section and equation numbers retain the prefix S. The local lemma index, placed after S5, links each named fact to its proof and principal uses. The printed contents give the section route; the PDF bookmarks retain the full subsection directory. The correspondence map in Section S12.7 identifies the formal interfaces for the same mathematical tasks. Each detailed component in S13–S15 begins with its scope, decisive obstruction and conclusion. S4 is the reference for common laws; the detailed sections prove their local consequences and exceptional cases.

Notation at a glance. The recurring counts are $W = e+R$, $R = \Pi+A+C+X$, and $m = |K|$. Here Π counts private columns and helpers, A alternating columns, C loop columns, X mixed columns, and K the centers with selected N columns. A canceled N aggregate still contributes to these counts. The coefficient $Z_{B,u}$ of a physical helper differs from its endpoint class $\zeta_{B,u} = D_B + Z_{B,u}$. The full notation table and the two incidence partitions appear in S4, before the index.

For the formal correspondence, this supplement's q counts supported Ω groups; Lean's budget variable `q` counts live null triples. The supplement scales retained coefficients by three from S4 onward; the corresponding Lean coefficients are unscaled. S12.7 gives the precise declaration map.

S2. The finite five-input proof

Let K be the binary kink matrix and B the degree-at-least-three Boolean content matrix. The lower system is $Kc = 0$, $Bc = e_{[5]}$. It is invariant under aggregating columns whose checked kink and Boolean data agree. The complete dictionary of 5,565 directions transfers to 3,245 stored columns with 2,626 distinct profiles: one zero profile and 2,625 nonzero profiles. This map is established for actual directions, rather than inferred from the sizes of the catalogues.

Lemma SL1 (The five-input profile facts)**.** *Let d, e be primitive, normalized directions in the complete five-input PM2 dictionary, and put $\tau(d) = \alpha_{[5]}(\rho(d \cdot m))$. Then $k(d) = 0$ implies $\tau(d) = 0$, and $k(d) = k(e)$ implies $\tau(d) = \tau(e)$.*

Finite proof and its formal justification. For a chamber π, let $g_\pi(d) \in \mathbb{Z}^4$ be the gap normal from the main paper. Every crossing normal of a primitive PM2 direction is one of the following vectors or its negative, in the displayed order:

$$\begin{array}{ccccc} (0,0,1,-1) & (0,1,-1,-1) & (0,1,-1,0) & (0,1,0,-1) & (0,1,1,-1) \\ (1,-1,-1,-1) & (1,-1,-1,0) & (1,-1,0,0) & (1,0,-1,-1) & (1,0,-1,0) \\ (1,0,0,-1) & (1,1,-1,-1) & (1,1,-1,0) & (1,1,0,-1) & (1,1,1,-1). \end{array}$$

Denote them by $v_1, \dots, v_{15}$. Define

$$k(d)_{\pi,j} = \mathbf{1}_{\{g_\pi(d) = v_j \text{ or } -v_j\}}, \qquad H_j(d) = \sum_{\pi \in S_5} k(d)_{\pi,j}.$$

The top coefficient is computed independently on the Boolean cube:

$$\tau(d) = \sum_{B \subseteq [5]} (-1)^{5-|B|} \rho\left(\sum_{a \in E_5} d_a m_a(\mathbf{1}_B) \right).$$

Enumeration of the 5,565 primitive directions gives 43 different histograms $H(d)$. The sixteen with nonzero top coefficient are listed below; every other attained histogram has top coefficient zero. Thus $\tau(d) = T(H(d))$ for this finite rule T, with $T(0) = 0$.

$(H_1, \ldots, H_{15})$	$T(H)$
$(0,0,0,4,0,0,0,0,4,0,4,0,0,4,0)$	1
$(0,0,0,8,0,0,0,0,0,0,2,0,0,4,0)$	1
$(0,0,0,8,0,0,0,0,4,0,6,0,0,8,0)$	1
$(0,0,0,12,0,0,0,0,0,0,0,0,0,0,0)$	1
$(0,0,0,16,0,0,0,0,0,0,0,0,0,0,0)$	2
$(0,0,0,16,0,0,0,0,0,0,4,0,0,8,0)$	2
$(0,0,0,16,0,0,0,0,4,0,8,0,0,12,0)$	2
$(2,0,0,16,2,0,0,0,0,0,0,0,0,4,2)$	2
$(4,0,0,6,2,0,0,0,0,0,2,0,0,4,0)$	1
$(4,0,0,10,2,0,0,0,0,0,0,0,0,0,0)$	1
$(4,0,0,16,4,0,0,0,0,0,0,0,0,8,4)$	2
$(6,0,0,10,4,0,0,0,0,0,0,0,0,4,2)$	1
$(8,0,0,8,4,0,0,0,0,0,0,0,0,0,0)$	1
$(8,2,0,8,6,0,0,0,2,0,4,2,0,6,4)$	1
$(8,4,0,0,4,0,0,0,0,0,0,0,0,0,0)$	-1
$(10,0,0,8,6,0,0,0,0,0,0,0,0,4,2)$	1

The enumeration can be reproduced without a stored profile catalogue. Generate all $15^4 = 50{,}625$ ordered choices of four feature slots, subtract their incidence vectors, remove zero, divide by the gcd and identify opposite signs. For each remaining direction, evaluate the 120 gap normals and the 32 Boolean inputs in the formulas above. The checker `code/check_max5_profile_facts.py` performs precisely these integer operations and verifies the rule, including the zero case. Its 3 November 2026 run passed on every direction and recovered the 91 coordinate/sign orbits independently; the receipt is `verification/max5-profile-facts-r12.json`.

The Lean proof supplies both coverage and the semantic link to these computations. The theorem `canonical_dictionary_complete` covers all nonzero PM2 directions up to scale. Checked permutation/sign tickets send every canonical direction to one of 91 shape representatives. Theorems `shapes_histogram_checked` and `shapes_top_checked` evaluate those representatives, and `shape_histogram_rule` checks their common rule. Histogram and top content are invariant under coordinate permutations and direction reversal; the latter changes a hinge only by a feature-linear term. Consequently `canonical_top_by_histogram` establishes $\tau = T \circ H$ on the complete canonical family.

Finally, a zero profile has histogram zero, while equal profiles have equal histograms. Applying T proves the two assertions. These are `canonical_zero_profile_top` and `canonical_equal_profile_top` in `PM2/ShapeInvariants.lean`. The checked quotient preserves both profile and top content, so the same assertions hold for its stored columns. □

The scope of this lemma matters. It concerns five-input PM2 directions, not arbitrary piecewise-linear functions or other dictionary sizes. For example, on three inputs, $\rho(m_{12} - x_3) = \mathrm{MAX}_3 - x_3$ has no interior chamber kink but has nonzero top content. Nor does the lemma identify all Boolean content: equal top coefficients need not give equal coefficients on proper subsets.

S2.1. **Circuit patterns and arbitrary rational scaling.** The main paper proves that every support-at-most-five solution contains a circuit with nonzero top content. Lemma SL1 excludes zero profiles and equal-profile differences as such an anchor, so this circuit has size 3, 4, or 5.

Three distinct nonzero binary columns have only the pattern $(1, 1, -1)$. For four, restriction to two independent rows could distinguish at most three nonzero binary columns, so a dependent four-column family has rank three. Its relation is unique up to scale. Separate positive and negative coefficients. In a one-versus-three split, the subsets of three positive weights summing to the opposing weight form an antichain. Rank three forces all singletons or all pairs, making the three weights equal. In a two-versus-two split, equal totals force all four weights equal; unequal totals force equal smaller-side weights and larger-side weights in the ratio $1 : 2$. The resulting four patterns are

$$(-1, 1, 1, 1), \quad (-2, 1, 1, 1), \quad (-1, -1, 1, 1), \quad (-2, -1, 1, 1).$$

For five-column circuits the bounded-cofactor argument gives coefficients in $\{\pm 1, \pm 2, \pm 3\}$. For each primitive candidate z, form all binary rows r with $r \cdot z = 0$. Their span must have rank four. Identifying the surviving vectors under permutation and sign gives the complete list:

1	$(-3, -2, 1, 1, 1)$	10	$(-2, -2, 1, 1, 1)$
2	$(-3, -2, 1, 1, 2)$	11	$(-2, -1, -1, -1, 2)$
3	$(-3, -1, -1, 1, 2)$	12	$(-2, -1, -1, 1, 1)$
4	$(-3, -1, -1, 2, 2)$	13	$(-2, -1, -1, 1, 2)$
5	$(-3, -1, 1, 1, 1)$	14	$(-2, -1, 1, 1, 1)$
6	$(-3, -1, 1, 2, 2)$	15	$(-2, 1, 1, 1, 1)$
7	$(-3, 1, 1, 1, 1)$	16	$(-1, -1, -1, -1, 1)$
8	$(-3, 1, 1, 1, 2)$	17	$(-1, -1, -1, 1, 1)$
9	$(-2, -2, -1, 1, 1)$		

This list classifies relations, not coefficient values in the representation. The common rational scale is unrestricted. Nor does every listed binary pattern necessarily carry nonzero top content in the actual PM2 dictionary.

S2.2. **Coverage of labeled anchors and extensions.** Stack K, B into A and set $y = (0, e_{[5]})$. Each proposed support J is tested by exact rational elimination or an equivalent projection. The negative witness is a row u satisfying $u^T A_J = 0$ and $u^T y \neq 0$. Multiplication then contradicts $A_J c = y$ without any coefficient bound.

Anchor	Complete extension family recorded in the formal proof
Five columns	6,765 reference anchors; no extra column.
Four columns	120 references, 960 pivot patterns, and 9,600 zero-profile extensions.
Three columns	145,800 unordered zero-extension pairs, including diagonal pairs; 146,025 actual projection memberships.
Grouped controls	117,960 groups and 40,530 within-group pairs.

Reference matching acts on the whole labeled support. Permuting an anchor and its extensions independently would not preserve coverage. Coarse grouping keys need not be injective: collisions add candidate comparisons and do not remove true relations. The final proof requires the verified membership and separation predicates, not these counts alone.

The declaration `attainable_stored_quotient_solution` connects an all-real PM2 identity to the stored equations. The circuit exclusion `stored_solution_support_gt_five` is joined to the proved reference coverage `canonical_saved_reference_coverage`. The final theorem roots are `PM2.WStar_eq_six` and `PM2.width_six_optimal`. S12 identifies their recorded checking procedure and complete source closure.

S3. Canonical reduction: exact finite obligations

The analytic touched-set and reconstruction proofs are in the main paper. This section describes the finite ingredients on which that transfer relies. They are logically separate from the later lower-bound case exclusions.

S3.1. **Generate the dictionary and the semantic rows.** Generate every primitive projective vector of the three forms $e_a - e_b$, $2e_a - e_b - e_c$, and $e_a + e_b - e_c - e_d$ in the lexicographic pair-feature coordinates. Choose a deterministic sign, retain loops, and compute touched sets after feature cancellation. For each direction and each of the 720 permutations, compute its gap vector from Boolean suffix values. Crossing normals supply the kink rows; Boolean Möbius inversion supplies the content rows. This defines the matrix semantically, independently of its stored row order.

The 1,980 covering directions have the following underlying patterns.

Pattern	Count
Three-edge star and disjoint edge	180
Four-vertex path and disjoint edge	540
Two disjoint three-vertex paths	270
Doubled edge in a perfect matching	45
Two disjoint edges and two isolated loops	135
Perfect matching and one loop	270
Three-vertex path, disjoint edge, isolated loop	540

Signs are included modulo simultaneous reversal. An independent reconstruction must compare actual rows and directions, rather than accept these totals as evidence of fidelity.

S3.2. **Compression invariant.** Initially each raw column is its own component with lifting weight one. At any stage a component variable z represents coordinates $c_j = e_j z$ with fixed nonzero rational e_j. Components have disjoint raw supports. A global one-entry equation sets a component to zero. A global two-entry equation joins two components with the forced nonzero rational ratio. Substituting in all rows preserves precisely the rational solution space. If a component survives with $z \neq 0$, its raw support cost is its number of members; different components cannot cancel each other's raw entries.

The completed trace removes 180 directions, leaves 1,530 singleton directions, and joins 270 raw directions into 90 triples. The checker must validate each sparse row at the time it is used; a row made sparse only by a prospective support choice is not a valid global step. The implementation is listed in Section S12.8.

S3.3. **Content of triples and retained sums.** For each triple, lift its component variable into the original columns. Exact substitution checks all kink rows, its star sum, and every Boolean coefficient. The required conclusions are closure, zero star sum, and four-set support at most one. Separately, row identities give, for every closed retained sum H,

$$\sum \text{star coefficients} = 2\alpha_{[6]}(H), \qquad \alpha_S(H) = 0 \quad (|S| = 3).$$

Together with star sum two, the center and correction rows give center mass $1/3$ and $\sum_{v \neq u} b_{uv} = 2/3$. The triple-content, canonical-row and weighted-model checks supply these identities; their entry points are in Section S12.8.

These are identities on the full retained solution space. Checking them on one upper construction would not suffice. A rational row-combination witness is checked by multiplying against every original column and the target.

S3.4. **Five-variable hosting.** The raw-touch-five family has 1,530 directions and 1,410 distinct nonzero kink profiles. Equal profiles have equal degree-four and degree-five content. After merging equal profiles in a closed zero-top block, only relations on three or four distinct profiles require computation.

For $A+B = C$, the two profiles on the left are disjoint. For $A+B = C+D$, enumerate unordered pairs and compare exact sums. For $2A + B = C + D$, A, B are disjoint, $C \cap D = A$, and $C \triangle D = B$. The remaining four-column patterns have analogous binary containment tests. Every retrieved candidate is checked by exact bit comparisons and then by its Boolean-content combination. Hash buckets retain all collisions, so a hash collision can only increase work.

Zero-top three-profile relations have at most one nonzero four-set coefficient; zero-top four-profile relations have at most three. Two independent implementations organize the same complete candidate family (Section S12.8). For five or more profiles the bound is immediate from the five available four-subsets. This proves the hosting inequality for every rational scale.

Support transfer. If the original identity has r retained terms, q_0 nonzero triples, and N_0 proper terms after compression, the number of required four-set corrections is at most $q_0 + N_0$. Its new width is therefore at most $r + q_0 + N_0 \le r + 3q_0 + N_0$. Max-basis reconstruction supplies the free term. The canonical matrix has 1,545 columns and 7,962 row coordinates; its equivalence to the original model uses all the preceding steps.

S4. The literal local model

All coefficients in this section are multiplied by three. Support counts are unchanged, each physical star-center mass is one, and the correction weights have signed degree two.

S4.1. **Nine families and their registry.** The retained dictionary is the union of the permutation orbits below, normalized modulo sign. This specifies the columns independently of IDs.

Family	Representative	Count
S	$m_{16} - m_{23} - m_{24} + m_{25}$	180
G	$m_{16} - m_{23} - m_{24} + m_{56}$	90
N	$m_{16} - m_{23} - m_{24} + m_{45}$	360
A	$m_{16} - m_{23} + m_{45} - m_{46}$	180
D	$m_{16} - 2m_{23} + m_{45}$	45
Z	$m_{16} - x_2 - m_{23} + m_{45}$	90
L	$m_{16} - x_2 - m_{34} + m_{45}$	360
E	$m_{16} - x_2 - x_3 + m_{45}$	45
X	$m_{16} - m_{23} + m_{35} - m_{46}$	180

The registry preserves each integer direction, family, physical center, private half/position, helper block, and the distinct alternating source-array and block-owner memberships. A permutation is applied to raw feature vectors before these labels are reconstructed. The fifteen fixed MAX4 correction representatives are returned to their chosen splits using feature-linear gauges; their raw list is not itself permutation-closed.

For a split $P \sqcup Q = [6]$ into triples, the glue at $(u, v) \in P \times Q$ compares the two stars of edges centered at u, v:

$$\sum_{p \in P \setminus \{u\}} m_{up} - \sum_{q \in Q \setminus \{v\}} m_{vq}.$$

The corresponding actual star replaces the second sum by $m_{uv} + m_{Q \setminus \{v\}}$. A row or column of the resulting glue matrix is a private line. Writing

$r = \sum_i l_i$, its equation is

$$S_i = l_i - r - \nu_i \qquad (i = 1, 2, 3). \tag{S4.1}$$

ν_i is an aggregate; its individual nonzero N summands are counted separately. They may cancel at a position.

S4.2. **Rows, blocks, arrays, and physical endpoints.** The literal homogeneous bank has 180 rows of each type T, V, h and 90 of each type S, g, a total of 720. A row is stored as a key and a sparse list of (column ID, integer coefficient) pairs. Section S12.8 identifies the literal row bank. Keys are labels for exact rows; their sparse entries, not a mnemonic interpretation of a letter, define the tested equation. For example, the archived zero-based row

$$S(0, 1, (2, 3), (4, 5)) : c_{210} + c_{450} + c_{725} + c_{1038} + c_{1392} + c_{1512} = 0$$

illustrates the literal convention. The upper-case row label S and the star family S are different uses of a letter.

A block $B = (G, U)$ consists of a perfect matching $G = \{U, V, W\}$ and a source $U = \{u, v\}$. Its helpers have arguments, up to orientation,

$$2m_U - m_V - m_W, \quad m_V + m_W - m_U - x_u, \quad x_u + x_v - m_V - m_W.$$

The diagonal vertex u is the physical Z attachment, and v is its mate. The coefficient is the stored hinge coefficient, unchanged by projective sign reversal. Set

$$\zeta_{B,u} = D_B + Z_{B,u}, \quad \beta_B = 2D_B + Z_{B,u} + Z_{B,v}, \quad \Gamma(u) = \sum_{B:u\in U} \zeta_{B,u} - \tfrac{1}{2}. \tag{S4.2}$$

Both h-side sums of a class at u equal $-\zeta_{B,u}$. A nonzero class needs at least one actual N on each side, on distinct lines. The total marginal at u is $\Gamma(u)$.

The array equations and block equations are

$$\begin{aligned} t^G_U &= \Omega_G + \beta_{(G,U)}, \qquad & w &= \sum_G \alpha_G \mathbf{1}_G, \\ \alpha_G &= -\Omega_G, \qquad & \sum_G \alpha_G &= 2, \end{aligned} \tag{S4.3}$$

$$\sum_{S \text{ in } B} S + \sum_{A \text{ in } B} A = \beta_B. \tag{S4.4}$$

An alternating has one owner in each partition. Their incidence lists must be reconstructed separately. In particular, a list obtained from the g block row is not automatically a list for an Ω source array.

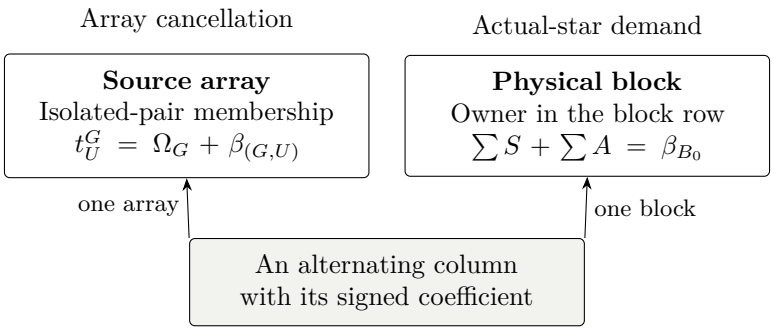


FIGURE 1. Two incidence partitions of the same alternating columns. Each column has one source-array membership and one physical-block owner, reconstructed separately from the literal dictionary. The owner B_0 need not be (G, U). Array and block arguments each give a lower bound on A; their counts can be added only after proving disjointness.

A worked derivation of the common array value. Take $G = \{12, 34, 56\}$, with coordinates numbered from one. Let r_i denote the homogeneous row with zero-based registry index i, and let c_j denote the coefficient of the column with zero-based ID j. The following vector identity uses full rows, before restricting support:

$$\begin{aligned}\mathcal{R} = {} & r_{212} - r_{233} + r_{242} - r_{263} \\ & + r_{390} + r_{391} - r_{411} - r_{412} + r_{420} + r_{421} - r_{441} - r_{442} \\ & - r_{540} - r_{543} + r_{648} + r_{651}.\end{aligned}$$

The first line consists of T rows, the second of V rows, and the third of h rows. Multiplication by the coefficient vector gives

$$\begin{aligned}\mathcal{R}c = {} & (c_{1138} + c_{1162} + c_{1231} + c_{1249}) - (c_{1289} + c_{1291} + c_{1295} + c_{1298}) \\ & - (c_{1266} + 2c_{1278} + c_{1284}) + (c_{1392} + 2c_{1437} + c_{1527}) \\ = {} & t_{34}^G - t_{12}^G - \beta_{(G,34)} + \beta_{(G,12)}.\end{aligned}$$

Every other column has coefficient zero in $\mathcal{R}$. Thus mixed, loop, and private columns have been canceled by row addition, not excluded from the allowed support. Since each input row annihilates every canonical identity, $\mathcal{R}c = 0$. The two corrected array sums are equal. Coordinate relabeling gives the comparison with the third source and the same conclusion for every matching.

This explains both parts of the definition: t sums the four actual alternatings in a source array, whereas β subtracts its doubled helper twice and each physical helper once. Their difference, rather than either sum separately, is constant across the three sources. An independent check multiplies all thirty archived comparisons over $\mathbb{Q}$ and verifies the displayed identity on all 1,545 columns (Section S12.8). In

Lean, `ContinuationArrayContentSemantics.array_equation` proves the common-value law for an actual canonical all-real representation. Its companion `omega_mass` gives $\sum_G \Omega_G = -2/3$ before the factor-three scaling used here, hence $\sum_G \Omega_G = -2$ in (S4.3).

S4.3. **Flow and projection as full-column identities.** Define

$$B_{uv} = -2 \sum_{\text{lines at } u \text{ containing } v} r, \qquad H^0_{uv} = (B_{uv} + B_{vu})/2. \tag{S4.5}$$

With side-flow orientation $V \to W$, the exact aggregate equations are

$$\operatorname{div}(\delta)_a = H^0_a + \sum_B \beta_B(\mathbf{1}_{a=U} + \tfrac{1}{2}\mathbf{1}_{a=V} + \tfrac{1}{2}\mathbf{1}_{a=W}) - w_a, \tag{S4.6}$$

$$2\sum_B \delta_B(\mathbf{1}_W - \mathbf{1}_V) = \sum_{B=(G,\{u,v\})} (Z_{B,v} - Z_{B,u})(e_u - e_v). \tag{S4.7}$$

The loop law is $2\delta_B = L_W - L_V$. A sole loop in a one-Z block has flow magnitude $|Z|/2$. The physical reconstruction checks fifteen edge-flow identities and six universal projection identities on all 1,545 columns, with six additional matching projection controls. Some input combinations use T rows with mixed entries; those entries cancel in the aggregate. Omitting them before combination would change the identity.

S4.4. **Costs and the pure normal form.** Write

$$R = \Pi + A + C + X, \quad \Pi = G + S + N + D + Z, \quad C = L + E, \quad h = d + z. \tag{S4.8}$$

For each line put $F = g + 2s + 2n$ and sum at a center to get $f(u)$. Then

$$2(\Pi - h) = \sum_u f(u), \qquad f(u) \geq 5 \ (u \notin K), \quad f(u) \geq 8 \ (u \in K), \tag{S4.9}$$

where K includes every center with selected N. A nonzero Γ at an active center gives $f \geq 9$. Since $A \geq 3q - h$, these imply

$$R \geq 15 + \lceil 3m/2 \rceil + 3q + C + X, \qquad m = |K|. \tag{S4.10}$$

Every helper belongs to only one β block, which is why at most h of the $3q$ arrays can be canceled, even with signed coefficients.

The pure move is $(D, Z_u, Z_v, E) \mapsto (D + a, Z_u - a, Z_v - a, E + a)$. This relation also has a direct functional explanation with consistent raw orientations. Set $K = m_V + m_W, \mathrm{D0} = 2m_U - K, \mathrm{E0} = x_u + x_v - K$, and $\mathrm{Z0}(u) = m_U + x_u - K, \mathrm{Z0}(v) = m_U + x_v - K$. Pointwise the unordered multisets $\{\mathrm{Z0}(u), \mathrm{Z0}(v)\}$ and $\{\mathrm{D0}, \mathrm{E0}\}$ agree: on $x_u \geq x_v$ the first Z is D0 and the second is E0, and on $x_v \geq x_u$ they swap. Their hinge combination is therefore zero. The stored quadruples have a common orientation in each block (15 positive and 30 negative); all 45 full-matrix relations are rechecked. An independent projective sign reversal adds only a feature-linear term, which is absorbed into $\ell \cdot m$.

Choose a lexicographic minimum of (R, z, d) at fixed correction data. If $Z_u = Z_v = a \neq 0$, the move with parameter a deletes two Z columns and

adds at most one D and one E. It cannot increase R and strictly decreases z, which is impossible in the chosen representative. If $D \neq 0$ and $Z_u = -D$, the move with parameter $-D$ deletes D and Z_u, adding at most one Z_v and one E. It cannot increase R or z, and decreases d if both are unchanged. Thus every surviving D has two nonzero endpoint classes, and Z occurs only at N centers. A block with one Z needs a loop. A block with two nonzero Z needs two loops: a sole E forces equal endpoint values, while a sole L occurs in only one of its two S rows. Summing over disjoint blocks gives

$$C \geq z. \tag{S4.11}$$

This bound is used only for the chosen lexicographic representative.

Reference for the local notation. Support counts are integers; the corresponding coefficients may be arbitrary signed rationals. The following distinctions apply in every case.

Symbol	Meaning and scope
W, e, R	Total width, correction support and retained support; $W = e + R$.
Π, A, C, X	Private and helper support, alternating support, loop support and mixed support; see (S4.8).
$K, m, f(u), h$	Centers with selected N columns, their number, the private line cost at u, and the helper count $d + z$. Balanced N still makes a center active.
$Z_{B,u}$; $\zeta_{B,u}$	A physical helper coefficient; the class value $D_B + Z_{B,u}$. A class can vanish while physical columns remain.
S; S^0	Actual star coefficients; effective stars $l - r\mathbf{1}$ obtained by eliminating N. The actual vector is $S = S^0 - \nu$. Support and block charges use actual stars unless stated otherwise.
t_U^G; β_B	The total in a source array; the helper coefficient $2D_B + Z_{B,u} + Z_{B,v}$. Source-array ownership and physical-block ownership are different partitions; see Figure 1.
Ω_G, q	The common corrected array value in matching G; $q = \#\{G : \Omega_G \neq 0\}$. Lean's budget variable `q` instead counts live null triples.
δ_B; c_B	Oriented block flow; its coordinate projection $\mathbf{1}_W - \mathbf{1}_V$. The projection law uses $2\delta_B$.
B_{uv}; H^0_{uv}	Directed glue-marginal content and its symmetric average (S4.5); H^0 is effective content.
L_3; z_B	In S13–S15, aliases for L and β_B, respectively. The count z is the number of nonzero physical Z columns.

Retained coefficients use the factor-three scaling from S4 onward; unscaled correction or Lean coefficients are identified explicitly. The coordinate labels in literal registry examples remain explicitly zero-based. Short equation

names such as (F) and (P) in the detailed cases refer to the corresponding laws of S4; their local specializations are stated at the point of use.

S4.5. **Matching-incidence rigidity.**

Lemma SL2. *A nonzero rational dependence among the fifteen perfect-matching incidence vectors on six vertices has at least six nonzero coefficients.*

Proof. Two distinct matchings share at most one edge. Each of a selected matching's three edges needs another selected matching, so at least four are needed. If three selected matchings share an edge, their other six edges are distinct. With five selected matchings, the other two must each meet all three once. The six equations make the first three coefficients equal and the last two their negatives; the common-edge equation forces zero. Four terms cannot even cover the six edges.

Otherwise each edge belongs to precisely two selected matchings. Joining matchings that share an edge gives a simple three-regular graph. On four vertices this is K_4, whose odd cycles force zero coefficients from the equations $a_i + a_j = 0$. On five vertices a three-regular graph is impossible by parity. This excludes support four and five. □

If the correction graph is M and $R \le 29$, a nonzero $\Omega + 2e_M$ would give $q \ge 5$ and then $R \ge 30$ by (S4.10). Hence $w = 2\mathbf{1}_M$ and $\Omega = -2e_M$ in that scope.

S5. Small local support lemmas

The following arguments explain the reductions from arbitrary rational coefficients to finite equality forms. They count faithful supports: a coefficient that vanishes belongs to a cheaper case.

S5.1. **Effective private lines and small layers.** With $N = 0$, equation (S4.1) is $S = l - r\mathbf{1}$. One glue gives two stars and cost five. Two nonzero glues a, b give stars $(-b, -a, -a-b)$, so their cost is six for $a+b = 0$ and eight otherwise. Three glues give costs five, seven, or nine if $r \neq 0$, and nine if $r = 0$. Thus

$$r \neq 0 : \ F \in \{5, 7, 8, 9\}, \qquad r = 0 : \ F \in \{0, 6, 9\}. \tag{S5.1}$$

Center mass one gives total marginal $-1/2$ at each center. If effective private support $P_0 \le 17$, there are exactly six nonzero-marginal lines, one per center, each with $r = -1/2$. The corresponding B row has two entries one and all others zero; nonempty zero-marginal lines make no contribution to B.

Lemma SL3 (The private-support gap)**.** *An effective N-free private layer with center masses one has $P_0 = 15$ or $P_0 \ge 18$.*

Proof. The six required nonzero-marginal lines cost at least 30, so $P_0 \geq 15$. If $P_0 \leq 17$, a seventh such line costs at least 35, and a nonempty zero-marginal line costs at least $30 + 6 = 36$. Neither fits. There are therefore exactly six lines, each with marginal $r = -1/2$. An F5 line of this marginal has only glue entries of magnitude $1/2$. An F7 line has three nonzero entries $(r, t, -t)$, up to permutation, where $|t| \neq 1/2$. Its two exceptional cells meet two distinct crossing lines, both of which must be non-F5.

For $P_0 = 16$, the excess over 30 is two, so there is one F7 and five F5 lines. Its exceptional cells give a contradiction. For $P_0 = 17$, the excess is four, leaving either one F9 or two F7 lines. An F9 line must contain a glue of magnitude different from $1/2$: otherwise its three signed entries of magnitude $1/2$ and sum $-1/2$ give two zero stars and make it F5. That exceptional glue would require a second non-F5 line, absent in the one-F9 case. In the two-F7 case, each F7 needs two distinct non-F5 crossing lines, but only one other exists. This excludes both supports. □

At $P_0 = 15$, all six lines are F5 and all effective stars are positive. If one line has three glues, its three crossing lines are nonempty. Its positive glue cannot meet a sparse F5 line of marginal $-1/2$; the resulting dense crossing line forces the remaining lines of the same split. The three positive cells form a permutation matrix, giving six stars in six distinct loop blocks. Otherwise three sparse glues give twelve stars, with at most two stars in any one loop block. The cores occupying exactly six blocks comprise 60 dense and 120 sparse forms. The other sparse forms occupy ten or twelve blocks; their incidences are reconstructed from the literal star columns. In particular, for an actual private-15 layer with zero helpers, (S4.4) requires

$$A \geq 6. \tag{S5.2}$$

This conclusion permits arbitrary loop and mixed columns, which cancel from the block-sum identity.

Eight-edge equality census. Suppose the actual private layer is N-free, all center masses are one, $P_0 = 15$, helpers and flow vanish, and exactly three matching groups support a correction graph with eight edges. The preceding F5 classification applies. Every edge weight is in $\{0, 1/2, 1\}$. Each of three distinct matchings has an edge unique to it, since each other matching shares at most one of its three edges. The three group weights are therefore positive, lie in $\{1/2, 1\}$, and sum to two; they are $1, 1/2, 1/2$ in some order.

A dense F5 form has all halves in one triple partition, so its H^0 has six edges, not eight. In a sparse form, the three glues pair the six centers. There are fifteen such pairings. For each paired u, v, choose the two halfmates of u among the other four vertices; the half at v is the complementary triple. These six choices for each pair give exactly $15 \cdot 6^3 = 3{,}240$ sparse forms. Each glue is $-1/2$, and each line has two stars $+1/2$.

For each form compute $B_{uv} = 1$ precisely when v is a halfmate of u, then $H^0_{uv} = (B_{uv} + B_{vu})/2$. Retain a form only if this vector has eight

nonzero entries and is the weighted sum of three distinct matching vectors with weights $1, 1/2, 1/2$. Exactly 360 labeled forms survive. They are the coordinate-permutation orbit of the following halves, listed in center order:

$$125, \quad 234, \quad 346, \quad 124, \quad 156, \quad 356.$$

Their halfmate pairs $25, 34, 46, 12, 16, 35$ are distinct. The two stars at a center have different block owners, so all twelve positive star demands belong to different physical blocks. With no helpers, the block equation forces $A \geq 12$. The independent integer enumeration is `code/check_outer_geometry.py`; its current receipt is `verification/outer-geometry-r20.json`. It checks this explicitly derived equality family, without imposing a grid on unconstrained rational coefficients.

At $P_0 \leq 19$, either the same six-line conclusion holds, or seven nonzero marginals cost 38: six F5 lines and one F8, without a zero line. One center has two lines. At least one meets only ordinary F5 partners, whose glue magnitudes are $1/2$; its marginal is therefore $\pm 1/2$. The two nonzero marginals sum to $-1/2$, so they are $1/2, -1$. Its B entries lie in $\{0, -1, 1, 2\}$, while the other rows are binary. Consequently

$$H_a^0 \leq 3/2. \tag{S5.3}$$

This is a signed bound. It is not a positivity claim at private support twenty; Section S8 treats that larger boundary separately.

S5.2. **Two nonzero classes at one center.** Two classes of values $a, b \neq 0$ require at least four N, two per class. Literal incidence gives two possibilities: equal source prefixes, whose shared positions are distinct, or distinct prefixes, whose line sets meet in one line. The latter main-pair choices have 2,160 three-line and 6,480 four-line incidences, checked on 540 class pairs. Equal-prefix position distinctions are checked on 90 pairs.

Lemma SL4 (Same-prefix cost-thirteen equality)**.** *At cost $f = 13$ with two same-prefix nonzero classes, there are exactly four N, no extra balanced N, no nonempty separating line, and $|a| = |b| = 1/2$.*

Proof. Six N leave no room for the required actual star. Five leave one glue and one star but occupy at least three lines, which cannot be serviced. Thus four N leave non-N cost five. An extra N-free line costs at least five and cannot fit. A singleton N line has non-N cost at least two, with equality only at marginal zero. For two distinct occupied positions on a shared line, direct solution of (S4.1) gives:

Non-N cost	Necessary relation or possible marginal
1	$a = b$, marginal a.
2	$a = -b$, marginal 0.
3	Three starless glues give $(a+b)/2$; one glue and one star give a or b.
4	Marginal in $\{0, a, b, a+b\}$; if $a = b$, the relevant values are $0, 2a$.

The starless vector is $((b-a)/2, (a-b)/2, (a+b)/2)$. Three occupied lines require costs $2+2+1$, giving $a = b = 1/2$ from $\Gamma = a + b - 1/2$. Two lines require $1+4$ or $2+3$. The first gives $a = b = \pm 1/2$; the second gives $a = -b = \pm 1/2$. Four lines cannot fit. □

For distinct prefixes at $f \leq 13$, five or more N and a four-line placement exceed the budget. With four N on three lines, distinct shared positions permit only extras $2+2+1$, giving $(a, b) = (1/2, 1/2)$. Coincident shared positions require $a + b = 0$; the shared aggregate is then empty and extras $2+3$ on the singleton lines give

$$(a, b) \in \{(1, -1), (-1, 1), (1/2, -1/2), (-1/2, 1/2)\}.$$

There is no lower-cost solution. In particular a value-two class together with another nonzero class cannot occur at $f \leq 13$ (T13). For $(a, b) = (2, -2)$, cost 14 also fails: the possible singleton marginal sums are $0, \pm 1, \pm 2$, never the required $-1/2$. Six N would need both a glue and an actual star, costing at least 15. This is the D13/14 argument and supersedes reliance on a bounded coefficient-mask test.

S5.3. **Unit classes and the G11 glue bound.** Suppose there is one nonzero class of value $+1$, actual star mass one, and $f \leq 11$. Its two h-side sums are -1 and $\Gamma = 1/2$. Three main N occupy three lines and cost at least twelve; five N and a star also cost twelve. Thus there are two or four N. Four leave exactly one glue and one star, so the glue is $1/2$.

With two N, each equals -1 on a distinct line. No additional N-free line fits. For one main line write $S = l - r\mathbf{1} + e_i$ and $t = g + 2s$. The two t values sum to at most seven. Solving the faithful support cases:

- $g = 0$ gives $t = 2, r = 0$.
- One off-i glue gives $t = 3, r = 1$ at value one; otherwise $t = 5$. A glue at i costs too much.
- Two glues at cost at most five give $l = -e_i + e_j$, $t = 4, r = 0$.
- Three starless glues give $l = (-1/2, 1/2, 1/2)$, $t = 3, r = 1/2$. With one actual star, $t = 5$ and its coefficient is $1 - 2r$.

Total marginal $1/2$ leaves only the splits $2+3$, $2+5$, and $3+4$. Every glue is in $\{0, \pm 1/2, \pm 1\}$. At exact cost eleven the last two splits give eight labeled forms. Negative actual stars in the $2+5$ case remain allowed.

An F5 separator with nonzero marginal r may contain N. By parity it has one or three glues. With no N, its glue vector is a single r or $(-r, r, r)$ up to permutation. With one N, the starless case has $\nu = -2re_i$ and $l = r\mathbf{1} - 2re_i$; with two N, only one glue fits. Thus every nonzero glue has magnitude $|r|$. A unit center with a separator at cost thirteen has line costs $4, 4, 5$ and separator marginal $1/2$. This includes balanced and coincident N where the incidence permits them.

Lemma SL5 (Together-line confinement, J12)**.** *Suppose a center has actual star mass one and exactly one nonzero class, of value one. If its line cost is $f \leq 12$, all its glues and actual stars lie on the two main together lines of that class. Additional balanced N are allowed. The same confinement holds for a sole class of value $1/2$ at cost $f \leq 10$.*

Proof. The two nonzero h-side sums require main N on two distinct together lines. Every line containing N costs at least four. For a unit class, five N and the required actual star use all twelve units, leaving no glue, contrary to $\Gamma = 1/2$. Three main N occupy three lines; four main N occupy four, while a main pair with a balanced pair may occupy two or more. Three N-lines at cost at most twelve would all be F4 and have zero marginal, again contradicting $\Gamma = 1/2$. Thus only the two main lines can contain N. An additional N-free nonempty line costs at least five, giving total cost at least thirteen. For a half class, three N-lines already cost twelve, and the same five-unit bound excludes an additional N-free line. This proves both claims. □

S5.4. **The half-class gap H11.**

Lemma SL6. *A center with a sole nonzero class $1/2$ and total marginal zero has no faithful support of cost eleven, even with additional balanced N.*

Proof. Three main N cost at least twelve. Four N at cost eleven leave one glue and one star; a sole nonzero glue contradicts total marginal zero. Thus two main $N = -1/2$ occupy two lines. A third nonempty line cannot fit. The line costs are $4 + 7$ or $5 + 6$. A one-N F4 or F6 line has marginal zero, while a one-N F5 line has marginal $1/2$ or $1/4$. A one-N F7 line cannot have zero marginal: its one-glue option is nonzero, and the three-glue zero-marginal option would require too many actual stars. Neither split has total marginal zero. □

S5.5. **Signed crossing arguments at the disjoint-source boundary.** At a unit/half-class private-26 boundary the center costs are $13, 9, 8, 8, 5, 5$. A separator at the cost-thirteen unit endpoint cannot be dense: it would force glue on a separating line at its unit mate. If sparse, its positive half glue cannot meet that mate or a glue-free half-class endpoint. It must meet an inactive F5 center, whose dense line then forces glue at a glue-free endpoint. This is again impossible.

An inactive F7 line has

$$l = (-1/2, t, -t), \qquad S = (0, t + 1/2, 1/2 - t), \qquad t \neq 0, \pm 1/2.$$

Its two exceptional glues reach distinct opposite centers. If their unit partners have cost at most eleven, G11 gives $|t| \leq 1$, so each actual star is at most $3/2$. Two such contributions cannot supply total mass four to two disjoint D-source blocks. If one unit center has cost thirteen, the excess budget forces all other unit centers to cost nine and all other inactive centers to cost five. Their glues have magnitude $1/2$, so both exceptional glues would have to meet the same high-cost center, impossible. The degeneracies $t = 0, \pm 1/2$ are cheaper F5 cases.

Local lemma and case index

This index records the scope, proof location and principal uses of the recurring local facts. The notation table and incidence diagram are in S4. Local-center entries assume actual star mass one; effective-layer entries state their center-mass hypothesis explicitly. Each name links to its proof. The entries are summaries, with the complete hypotheses and argument at the linked location. Numbered lemmas have labels SL1, SL2,. . . , independently of sections and equations.

Name	Scoped statement	Proof	Principal uses
Matching independence	A nonzero relation among perfect-matching incidence vectors uses at least six vectors.	Lemma SL2, p. 14	S13.5, D
Line spectrum	Effective N-free lines have costs $\{5,7,8,9\}$ at nonzero marginal and $\{0,6,9\}$ at zero marginal.	S5, p. 14	S6, S8
Private gap	An effective N-free layer with all center masses one has $P_0 = 15$ or $P_0 \geq 18$.	Lemma SL3, p. 14	S6, S13.4
Private 15	For an actual private-15 layer with zero helpers, the block demands force $A \geq 6$.	S5, p. 15	S13.4, S13.5
P19	For an effective N-free layer with center masses one and $P_0 \leq 19$, every $H^0_e \leq 3/2$.	S5, p. 16	S7.1, S13.5, C
Same-prefix 13	Two nonzero same-prefix classes at $f = 13$ have exactly four N, no separator, and $\lvert a\rvert = \lvert b\rvert = 1/2$.	Lemma SL4, p. 16	S9, S14.2
D13/14	Two distinct-prefix classes of values $(2,-2)$ cannot have $f \leq 14$; balanced N are included.	S5, p. 17	S15.7
G11	A sole unit class at $f \leq 11$ has glue values in $\{0, \pm 1/2, \pm 1\}$; signed stars are allowed.	S5, p. 17	S15.3
J12	A sole unit class at $f \leq 12$, or half class at $f \leq 10$, has all glues and stars on its two together lines.	Lemma SL5, p. 18	S15.2, S15.3
H11	A sole half class with total marginal zero cannot have $f = 11$, even with balanced N.	Lemma SL6, p. 18	S10, S15.6
H / HA	Every normalized identity satisfies $R \geq 15 + \lceil 3m/2\rceil + 3q + C + X$ by disjoint array charges.	S13.5, p. 100	S7, S13.5, D
O29	With matching corrections and $R \leq 29$, matching independence and HA force $\Omega = -2e_M$.	S13.5, p. 105	S8, S9, S10
C18 / erasure B	At $P_0 \leq 18$, center masses one and unit six-cycle H^0 force the positive private-15 core; N can erase at most one charged block per column.	S13.5, p. 103	S13.5, C, S7.2
P24	An effective N-free layer with center masses one and $H^0 = 2M$ has $P_0 \geq 24$.	S13.1, p. 69	S13.2, S13.4
P25	Under the same effective-layer hypotheses, the nine-line argument strengthens the bound to $P_0 \geq 25$.	S13.5, p. 109	S13.5, F
P27	An effective N-free layer with center masses one and $H^0 = 2M$ has $P_0 \geq 27$.	Lemma SL7, p. 32	S8
$L1^+$	One nonzero physical Z globally requires $C \geq 6$; if its block has one loop, $C \geq 7$.	S13.5, p. 112	S14.8, S15.1

Name	Scoped statement	Proof	Principal uses
Two-class S	Two distinct nonzero helper classes at one center force $f \geq 13$, including same-source classes and balanced collisions.	S13.5, p. 120	S14.1, S14.6
One-block Z	Exactly two global Z in one block, with $C \leq 5$, have equal coefficients and zero flow.	S13.5, p. 121	S14.6, S15.1
B9	An active center with all helper classes zero and $f = 9$ has nonnegative stars, binary B, and half-magnitude nonzero glues.	S14.1, p. 131	S15.5, S15.8
DC / D5	For $Z = 0$, repeated D incidence gives $\Pi \geq 18 + 2m$. At $m = 5$, $\Pi \leq 27$ forces disjoint sources and $\Pi \geq 25 + t$, where t counts nonhalf D.	S14.1, p. 132	S14.2, S15.5
E23	With matching corrections, exactly one D, zero flow and all other classes zero, its endpoint costs sum to at least 23.	S14.5, p. 145	S9.2, S15.3
U25	In the same zero-flow setting, a unit D on a matching source has endpoint cost sum at least 25.	S14.4, p. 140	E23, S15.3
B10	An active center with all classes zero and $f \leq 10$ has $f = 9$ or 10, nonnegative stars, and positive glues at most 1/2.	S14.5, p. 146	S9.2, S15.5
P14	Two distinct-prefix classes of values $(1/2, 1/2)$, $(1, 1/2)$ or $(1, 1)$ at $f \leq 14$ have nonnegative marginals and $B \leq 0$.	S14.5, p. 146	S15.2
SZ6	For one nonzero Z globally and exactly $C = 6$, flow divergence at its source is zero; other direction families remain allowed.	S14.8, p. 163	S15.1, S15.7
Z13	A sole value-two class cannot have $f \leq 13$ when every glue is in $\{0, \pm 1/2\}$; balanced N are allowed.	S14.8, p. 164	S15.7
T13	Two distinct nonzero classes at $f \leq 13$ cannot include a value-two class; balanced N do not evade the bound.	S14.8, p. 164	S15.1, S15.8
O15	Two opposite Z at one attachment, with distinct sources, one matching source, binary inactive B, and flow only in those blocks, cannot have $f \leq 15$.	S14.8, p. 165	S14.8, S15.7
N6	At the $R \leq 28$ one-attachment boundary $(\Pi, A, C, X) = (23, 3, 2, 0)$, six N are excluded with every actual-star position allowed.	S14.8, p. 168	S6.4
B11 / parity	A helper-free active center at $f \leq 11$, with the other two active centers glue-free and three inactive F5 centers, has $B \leq 2$; half-grid glues also give $f \equiv 2\Gamma \pmod 2$.	S15.8, p. 198	S15.8
HE5	Five correction edges under $R \leq 27$ are impossible, by signed graph, private-cost and Hall-witness arguments.	S7.1, p. 27	S6.4, S7
HE6	Six correction edges with two supported matching groups under $R \leq 26$ are impossible.	S7.2, p. 29	S6.4, S7
H32	The preliminary chain, retained-28 matching cover and independent nonmatching bounds exclude width 31.	S6.4, p. 25	S8, S9, S10

Name	Scoped statement	Proof	Principal uses
D21	The literal twenty-one-row combination on the full 403-column scope gives $-D + t/2 = 0$, excluding the unit-D boundary.	Lemma SL8, p. 71	S9.2
SDX1	The single-D core with any one mixed column is excluded by exact row identities covering all 180 choices.	S9.2, p. 36	S9.2
SD2A	For unit $D(M, U)$, the 403-column core plus any of four allowed outside arrays gives a 407-column inconsistent system.	S9.2, p. 36	S9.2
SD3M	Adding all N at any one of four further centers gives four inconsistent 463-column single-D spaces.	S9.2, p. 36	S9.2
R-SD-L10	The outside-group single-D case is excluded on all three 407-column geometries, without assuming inactive cost-ten lightness.	S9.2, p. 36	S9.2
Sole Z, $m = 2, 3$	At matching $R \leq 29$, a representative minimizing (R, z, d) cannot have exactly one Z and two or three attachments.	Lemmas SL9–SL10	S15.6, S15.8
Helper counts	At matching $R \leq 29$, $m = 2$ and $Z > 0$, only $(d, z) = (0, 2), (0, 3), (1, 2)$ remain in the lexicographic minimum.	Lemma SL11, p. 171	S15.6
Same-block pair	At matching $R \leq 29$ with $m = 2$ or 3, exactly two global Z cannot share a block in the lexicographic minimum.	Lemma SL12, p. 171	S15.6, S15.8
Six attachments	No matching identity with $R \leq 29$ has six active attachments, including helper-free balanced attachments.	Lemma SL13, p. 172	S8, S10
Two-Z arrays	For matching $R \leq 29$, $m = 2, d = 0, z = 2$, distinct attachments in one matching and one outside block, and no third-block flow, only two $X = 0$ affine corners remain.	S15.6, p. 186	S10

S6. The preliminary bound and exclusion of width thirty-one

The width-31 exclusion is retained as a separate stage because later normalizations use it. This section identifies its foundation, gives the zero-helper proof, derives the source-degree reductions, and specifies the remaining component cover. The full preliminary chain is proved in Section S13; the nonzero-helper components are given in Section S14.

S6.1. **The complete retained-cap-twenty-one ingredient.**

The necessary-support input has 1,530 directions, 3,951 homogeneous row supports, and 18 mandatory groups. Six groups are star-center groups; the other two families are transport and P4 groups. The homogeneous rows include certified three-set and edge-consistency equations derived from feature incidences. Their coefficients may be signed or have magnitude two; the support search uses only the necessary fact that a homogeneous equation cannot contain exactly one selected nonzero coefficient.

The groups are necessary because exact row identities give nonzero target sums on them. For instance, with $B(v, S; d)$ the incidence of one positive and one negative item at v whose remaining union is S, the P4 identity is

$$\sum_{d} c_d \sum_{|S|=4,\, v\notin S} B(v, S; d) + 2C_v = 0.$$

Since the unscaled center mass is $C_v = 1/3$, each P4 group must be met. Each selected P4 column meets at most two such groups, and these groups are disjoint from the star and transport families in the required counting sense. The additional completion bound is

$$\#\text{missing stars} + \lceil \#\text{missing transport}/2 \rceil + \lceil \#\text{missing P4}/2 \rceil.$$

The input export is checked against literal equations, including all 75 added translated equations and 9,180 incidence equalities.

The completion-partition algorithm works as follows.

(1) Fix a star anchor after applying the proved coordinate symmetry. Track selected and excluded columns, occupied rows, singleton rows, and groups already met.
(2) Every singleton row requires one further column incident to it. Every missing group requires a column in that group. Remove unavailable columns from these requirements. An empty requirement is impossible.
(3) A collection of disjoint requirements needs at least that many new columns. Prune only when this number or the certified group bound exceeds the remaining support budget.
(4) Choose one ordered requirement. Branch on its possible members; after taking one member, exclude its earlier siblings in later branches. Every completion has a unique first chosen member of that requirement.
(5) If no requirement remains, record the closed support and still branch over every unselected, unexcluded column. This includes disconnected

extensions. With one slot left, retain only choices meeting all requirements and creating no new singleton row.

Induction on remaining slots proves exhaustive coverage. Pruning uses necessary support conditions only. No numerical row-space test or assumed minimum-circuit structure is used. Timeout, node, or memory limits must report an incomplete run.

The implementation and its certified incidence input are listed in Section S12.8. The older six-group center search is not the eighteen-group cap-21 program. The fresh complete partition run visited 16,008,595 nodes and found no closed support. The separately archived binary include/exclude runs on the two exhaustive anchored helper cases visited 8,479,435 and 2,330,545 nodes, again with no closed support. These runs share a certified necessary input but use different search logic. Their conclusion is retained support $R \geq 22$, not yet the final global lower bound.

S6.2. **The preliminary chain.** The preliminary chain uses the preceding ingredient and local equations to successively rule out the following boundaries:

Result	New boundaries and resulting retained estimates
$W \geq 27$	Matching $R \leq 23$ excluded; nonmatching $R \geq 22$ has at least five corrections.
$W \geq 28$	Matching $R \leq 24$ excluded; five-edge graphs require $R \geq 24$.
$W \geq 29$	The width-28 boundaries $(e, R) = (3, 25), (6, 22)$ are excluded.
$W \geq 30$	The width-29 boundaries are $(3, 26), (5, 24), (6, 23), (7, 22)$.
$W \geq 31$	The width-30 boundaries are $(3, 27), (5, 25), (6, 24)$.

The final retained bounds are 28 for a matching, 26 for five edges, 25 for six edges, and 24 for at least seven edges. Signed degree excludes fewer than three edges and exactly four. Their total-width minimum is 31.

Each step is proved in full in Sections S13.1–S13.5, including its normalization, local equality cases, exhaustive case split, and terminal equations. The literal model uses the physical centers of all 360 nonalternating, 180 star, 90 diagonal-helper, and 45 doubled-helper columns. Its center, side, array, block, loop-difference, source, flow, and projection equations have been checked coefficient by coefficient.

Two scope details prevent a false extension of this foundation. The fixed 21-row dual used in the early D cases has 24 nonzero mixed residuals; its 399- and 403-column uses require $C = X = 0$. Positive-X cases have separate arguments. In the earlier one-in/one-out Z case, the budgets force the queried active mate into the proved cost-at-most-ten range; the later cost-eleven counterexample is outside those hypotheses.

S6.3. **Zero helpers at retained support twenty-eight.** Suppose $w = \mathbf{21}_M$ and all normalized helpers vanish. Absorb N into effective stars using (S4.1). This preserves center masses and gives $P_0 \leq \Pi$.

First, an effective layer with $H^0 = 2M$ has $P_0 \geq 26$. To prove this, assume $P_0 \leq 25$. Each matching edge needs a nonsimple endpoint, giving at least nine nonzero-marginal lines. With nine, three centers are simple and their mates have masses $a, 1-a$ with $a \in \{3, 4\}$. Their directed rows lack the value -1, but a simple center's offmatching entry one needs a reciprocal -1. Ten lines force ten F5 lines and no others. The distribution $(3, 2, 2, 1, 1, 1)$ has the same obstruction. In the remaining distribution $(2, 2, 2, 2, 1, 1)$, let the simple centers be s, t, their mates p, r, and the last matched pair q, v. Both q, v must have masses $2, -1$, because a sole negative-mass line at one would have no compatible opposite glue partner. The positive masses of p, r must agree in $\{3, 4\}$ and their halves be complementary. The matching and offmatching equations then leave each negative p half with no complementary line of the required glue magnitude. This excludes $P_0 = 25$.

At zero flow, $H^0 = 2M$, so $R \geq P_0 + A \geq 26 + 3 = 29$. At nonzero flow, $C \geq 6$, $A \geq 3$, hence $\Pi \leq 19$ and $m \leq 2$. If $m = 0$, private support is 15, 18, or 19; the private-15 case needs $A \geq 6$. If $m = 1, \Pi = 17$, its active cost is nine with exactly two balanced N, so at least four of the six positive blocks remain demanded. All other cases have $\Pi \geq 18$. Thus $C \leq 7$ throughout, leaving exactly three flowing blocks.

Their divergence has at most six nonzero pair nodes. By (S4.7), H^0 still has degree two at every vertex. The bound $H^0_a \leq 3/2$ makes all three matching edges negative-demand nodes for $H^0 - 2M$. Positive offmatching edges cover all six vertices, so there are exactly three of them and no other demand nodes. They form a disjoint matching M'. The two matchings form a six-cycle, and simple-simple edges force both alternating weights to be one.

At $P_0 \leq 19$, a unit six-cycle has exactly its private-15 core. Seven nonzero marginals would put two distinct halves at one center, but the three cycle nonneighbors force both halves to equal its ordinary cycle half. With six nonzero marginals, any nonempty zero line forces a second, exceeding the budget. Thus three glues and twelve positive stars remain, in twelve distinct blocks. The possible balanced N affect at most four blocks; hence $A \geq 8$ when $m > 0$, and $A \geq 12$ when $m = 0$. Either contradicts $R \leq 28$.

S6.4. **Nonzero helpers: complete H32 routing.** With $Z = 0$, every nonzero D has two active endpoints, so $m \geq 2$. For source degree r_u, the charge

$$\pi(u) = f(u)/2 + r_u/2,$$
$$J(0) = J(1) = 9/2, \quad J(2) = 15/2, \quad J(r) \geq 1 + 5r/2 \ (r \geq 3)$$

applies at active centers; inactive centers cost at least $5/2$. At $m = 2$, all D have the same source. Two require $\Pi \geq 25$ and $A \geq 3$; equality at $R = 28$ has two cost-thirteen endpoints and cannot supply the required separating sum $NT = 1$. Three or more give $\Pi \geq 12 + 5d$, $A \geq d + 1$, hence $R \geq 31$.

At $m = 3$, two repeated sources cost $\Pi \geq 27, A \geq 3$. Three-edge multigraph degree types $(2,2,2), (3,2,1), (3,3,0)$ give $\Pi \geq 30, 28, 29$, respectively, before the two unchanged arrays. Only one D or a two-edge path survives. At $m = 4$, two high-degree vertices cost at least 29; two intersecting sources give $\Pi \geq 26, A \geq 3$. The remaining higher-edge graph is a three-edge star, with $\Pi \geq 27, A \geq 4$. Only one D or two disjoint sources survives.

The remaining component cover is:

Hypotheses	Terminal argument and finite inputs
$Z = 0$, one D	Separate zero and nonzero flow, source in or outside M, and unit/nonunit values. Source balance and private costs; the single-D checks include 360 balanced bridges and 540 alternating owners.
$Z = 0$, $m = 3$ path	Endpoint source balance and the positive two-class cost bound; no repeated-source case is omitted.
$Z = 0$, $m = 4$ disjoint	The private-25, 26, and 27 boundaries; source demands and signed star supply.
$Z > 0$, $m = 1$	The $R \leq 28$ one/two-class exclusions in Section S14.8, including O15 and the N6 argument with every actual-star position.
$Z > 0$, $m = 2$	Physical source/side/absent projection, same or distinct attachments, and both helper locations. The shared-line argument below is used.
$Z > 0$, $m = 3, 4$	Cancellation counts followed by the explicit outside-helper cost exclusions below.
$m = 5, 6$	Private costs, helper degree, and array charges, including helper-free balanced active centers.

Sections S14.1–S14.8 give the full component proofs. S11 specifies their exact finite predicates. Their hypotheses partition all nonzero-helper identities; zero helpers were handled before this partition.

The shared-line partner argument. A marginal $-3/2$ on an isolated shared line can have cost seven; it cannot be ruled out by a generic cost-eight assertion. In the actual two-line case the entries are $(-b, t)$ and $(-b, -t)$ in distinct positions, $b = \pm 1/2$, with total cost at most twelve. If the target line costs seven, its partner must cost five. The partner's F5 law forces $t = b$ after orientation. The target then has opposite coefficients of magnitude $1/2$, whose F7 marginal is $\pm 1/2$, not $-3/2$. Cost at least eight would put the pair above twelve. This includes arbitrary balanced t.

Outside-helper allocation costs. At three attachments, an M-Z of value two plus an outside Z with $A = 3$ has active costs at least $11, 8, 9$ and inactive cost 15. Thus $\Pi \geq 24$ and $R \geq 24 + 3 + 2 = 29$; it is not a retained-28 cell. The surviving $A \geq 4$ case forces $(\Pi, A, C, X) = (22, 4, 2, 0)$ with two half-class endpoints, a helper-free cost-nine center, and inactive F5 centers. Its physical projection is tested in precisely that scope. At the putative $(24, 2, 2, 0)$ three-helper boundary, two value-two matching cancellations plus

one outside Z give $\Pi \geq 26$ if their attachments differ, or $\Pi \geq 27$ if the outside Z shares an attachment. Only after these exclusions may all helpers be placed in M.

S6.5. **The H32 join.** The bound $W \geq 31$ leaves exactly four total-width-31 budgets: matching $(3, 28)$ and nonmatching $(5, 26), (6, 25), (7, 24)$. The matching cover above excludes the first. The stronger nonmatching bounds in S7 and the seven-edge argument in the main paper exclude the other three; the six-cycle step uses the private-at-most-nineteen lemma only in its unit-cycle scope. For $e \geq 8$, $q \geq 3$ and (S4.10) give $R \geq 24$, already too large. Any strict support decrease at this stage contradicts only $W \geq 31$. This establishes H32 without a matching-$R = 29$ premise.

S7. Nonmatching corrections under the width-thirty-two budget

Assume an actual normalized identity with e nonzero correction edges and retained support $R \leq 32 - e$. No lower bound on its total width is assumed. We prove that $e = 5$ requires $R \geq 28$ and $e = 6$ requires $R \geq 27$. The arguments use the local laws of S4–S5 and the stated support budgets; they do not use H32 or a matching-$R = 29$ exclusion. Consequently they also exclude the smaller budgets $(e, R) = (5, 26), (6, 25)$ in the H32 join. The main paper supplies the signed correction census and the cases with at least seven edges.

S7.1. **Five correction edges: HE5.** Here $R \leq 27$. The signed graph census has two possible geometries: an isolated weight-two edge plus a four-cycle, or a connected double-star with four leaf weights two and central weight -2. The second cannot be silently dropped.

Derive $q = 2$. Equation (S4.10) rules out $q \geq 5$. At $q = 4$, only $m = 0, \Pi = 15, A = 12, C = X = 0$ fits. Its private rows have $H^0 \leq 1$, contradicting the required weight-two edge. At $q = 3$, the $m = 0, 1$ cases have no helper or flow and $P_0 \leq 18$; (S5.3) gives the same contradiction. At $m = 2$, $C = X = 0$. No D gives the same private contradiction; two or more exceed the budget. A single D forces $\Pi = 19, A = 8$, half-class endpoints of cost eight, and inactive F5 centers. Equation (S4.6) makes $w \geq 0$. Each of the three matchings has a unique edge, hence positive weight, so their union has at least seven edges. Therefore $q = 2$.

The two matchings share exactly one edge, with weight two, and their other edges form a four-cycle with nonzero alternating weights $\alpha, 2-\alpha$. Signs remain unrestricted. The helper-array bound becomes $R \geq 21 + \lceil 3m/2 \rceil + C + X$, so $m \leq 4$.

Zero helpers. At zero flow the effective H^0 has an isolated weight-two edge. There must be at least nine nonzero-marginal lines: with at most eight, a simple endpoint needs an offmatching reciprocal -1 which cannot be supplied consistently by at most two doubled centers. Nine F5 lines cost 45; the

spectrum permits no total 46. Thus $P_0 \geq 24$, while $A \geq 6$ leaves $P_0 \leq 21$. Nonzero flow forces $(\Pi, A, C, X) = (15, 6, 6, 0)$ and $m = 0$. At private fifteen each center has one F5 line. A dense line forces all six into one split, giving six signed permutation matrices per split, or 60 dense cores. Otherwise three sparse glues pair the centers. The 216 generator choices for each of fifteen pairings have 6, 10, or 12 block owners; exactly eight per pairing have six, giving 120 sparse cores.

Each of six nonzero source arrays must choose an alternating in one of six demanded blocks, with one per block. Construct the bipartite incidence graph from the actual columns. Of the 180 cores, 148 have an array with no eligible block and 32 have two arrays with only one eligible block. These exact Hall witnesses exclude every core. This family is different from the capacity-two matching family in S8.

Z present. A sole Z is too expensive in loops. At $m = 1, z \geq 3$, the class and array counts give $R \geq 20 + 3z \geq 29$. At $m = 1, z = 2$, equality forces $\Pi = 21, A = 4, C = 2$, an active cost-thirteen center and five inactive F5 centers. Both helper arrays must cancel. Their class values sum to two and have equal magnitude, so both are one. The active center then needs three positive half glues and four $N = -1$. Its effective stars are integral, forcing all three glues into one line; that line needs at least three N, but the two classes supply at most two on one line.

At $m = 2$, a second class at either center or three helpers already exceeds the budget. Two Z in one block force $\Pi = 20, A = 5, C = 2$: half-class endpoints, unit matching weights, and zero projected flow. On the shared weight-two edge the required H^0 is $3/2$, but inactive rows give at most one. In distinct blocks, the private-20 and private-21 cases are separated. If both cancel in one group, the flow is constant on its pairs while the two endpoint charges are antisymmetric, impossible. If only one cancels at private twenty, group weights $1/2, 3/2$ leave an edge whose helper-plus-private contribution is too small. At private twenty-one, both cancel in different groups, with unit values. Projection forces one source to join the two active vertices. Its required $H^0 = 0$ contradicts one directed entry -1 and a nonpositive reciprocal. For $m = 3, 4$, the loop budget is at most one or zero, excluding normalized Z.

D only. Nonzero flow is too expensive. At $m = 2$ at most one D survives, with $\Pi \leq 22$. An unsupported group leaves $A \geq 7$ and cannot supply the isolated weight-two edge. In a supported group, $\Pi \leq 21$ forces the unit case and an adjacent weight-one edge with $H^0 \leq 1/2$. At $\Pi = 22$, equality gives $A = 5, C = X = 0$. Source balance removes the common-edge source and forces a half-class on a unique cycle edge. The H11 gap prevents both endpoint costs being eleven, so one has $B = 0$; a cheap adjacent inactive row again cannot supply weight one.

At $m = 3$, repeated or multiple sources exceed the degree budget, leaving one half-class D. A helper-free active center is charged at cost nine; one of

the two other cycle neighbors remains inactive, giving the same adjacent-edge contradiction. At $m = 4$, one D or intersecting sources already cost too much. Two disjoint sources force half-class endpoints, $\Pi = 23, A = 4$, and two canceled arrays. Whether they lie in the same matching or different matchings, one source receives helper contribution $3/2$ where its correction weight is one or two. Higher degrees are excluded by the monotone source-degree charge.

This covers zero helpers and every possible nonzero-helper attachment count. The finite controls reconstruct the Hall witnesses, source incidences, and faithful H11 masks; the signed reductions above establish their scope. Thus no five-edge identity has $R \le 27$, proving $R \ge 28$.

S7.2. **Six correction edges with two groups: HE6.** The $q = 3$ case is excluded in the main paper by the small private-layer argument. With $q = 2$, the two matchings are disjoint and form a six-cycle with alternating weights $\alpha, 2-\alpha$, initially of either sign. Now $R \le 26$ and $R \ge 21 + \lceil 3m/2 \rceil + C + X$, so $m \le 3$.

A sole Z cannot fit its loop charge. At $m = 1$, two classes give $\Pi \ge 21, A \ge 4, C \ge 2$, already 27; more helpers cost more. At $m = 2$, two classes at a center or three helpers are similarly too expensive. Two Z in one block leave five arrays and cost at least $20 + 5 + 2 = 27$. Distinct blocks force $\Pi = 20, A = 4, C = 2$, with both values $1/2$ and both arrays canceled. They must belong to the same matching, since group weights sum to two. Its pair-constant projected flows cannot cancel lone antisymmetric endpoint charges. At $m = 3$ no loops fit. Thus $Z = 0$; nonzero flow costs six and is also excluded.

For D, the three-attachment degree types remove repeated and multiple sources. A lone D has half-class endpoints with zero B rows; source balance forces cycle weights one, but another cycle edge at an endpoint has $H^0 \le 1/2$. With two attachments, two D cost at least $\Pi = 25, A = 4$; three or more cost still more. One D has $\Pi \in \{19, 20, 21\}$. At private nineteen the same half-class contradiction holds. At private twenty a half-class again has zero rows; a unit class instead forces the two dense active lines and all inactive lines into one split, whose row marginal sum $1/2$ differs from its column sum $-3/2$. At private twenty-one, the one canceled array and vanishing separating sums force a half-class; some endpoint costs at most ten, and the adjacent-edge contradiction applies.

It remains $D = Z = 0$, zero flow, $A \ge 6$, and $P_0 \le \Pi \le 20$. There are at most eight nonzero-marginal lines, hence at most two nonsimple centers. Each cycle matching has a simple-simple edge, forcing $\alpha \le 1$ and $2 - \alpha \le 1$, so $\alpha = 1$.

Six lines give the private-15 cycle core; a zero line would force another and cost too much. Seven lines are impossible by the nonneighbor argument in S6. With eight, every line is F5. A tripled center has only three possible complementary halves; the zero-edge equations kill two, so it cannot be

tripled. Two doubled centers must each have masses $2, -1$: an exclusive simple halfmate restricts the possible mass to $\{2, 1, 0, -1\}$, and zero or one would collapse a line. Their positive lines can glue only to one another, making their halves complementary and their centers opposite on the cycle.

In zero-based cycle order 012345, take doubled centers $0, 3$. Their positive halves are $015, 234$. The negative 0 half is $0 + S$ for a two-subset $S \subset \{1, 2, 4, 5\}$ other than $\{1, 5\}$. The negative 3 half is its complementary choice; every simple half is then forced. There are exactly five half configurations, not a guessed set of coefficient values.

If the effective layer is private fifteen, its twelve positive block sums can be erased only by positive N. At each balanced active center at least one N is negative. Hence

$$A \geq 12 + m - N, \qquad N \leq \Pi - 15 + m, \qquad \Pi + A \geq 27,$$

contradicting $R \leq 26$. At private twenty equality forces $\Pi = 20, A = 6, C = X = 0$. Cellwise equality in the support inequality means every effective star is replaced by precisely one actual star or one N with the same value. Simple centers cannot have balanced erasures. At a doubled center only a dense negative line permits an erasure: a $+1$ star paired with a -1 star in the same balanced class and side. Sparse negative lines would need an odd three-term erasure, forbidden by the side pairing.

The five half configurations yield 33 private layers and 47 actual forms: 33 with no active erasure, 12 with one, and 2 with two. Only two have the nonnegative integral block sums required for one coefficient--1 alternating in each of the six arrays. Literal incidence forces their unique completions:

Negative 0 half	Alternating column IDs, each coefficient -1
012	$55, 200, 237, 1138, 1224, 1295$
045	$37, 51, 805, 1097, 1231, 1289$

Each fails eight literal T rows, four with residual $+1$ and four with -1. For example, their residuals at $T(0; 12; 3; 45)$ and $T(0; 34; 5; 12)$ are respectively $+1$ and -1. The six-edge checker (Section S12.8) reconstructs the forms, erasures, owners, and all sixteen residuals. Thus no six-edge identity has $R \leq 26$, proving $R \geq 27$.

S8. Matching corrections: the envelope and zero helpers

Fix $M = 12 \mid 34 \mid 56$. Matching-incidence rigidity gives $w = 2\mathbf{1}_M$, $\Omega = -2e_M$ at $R \leq 29$. The normal form and local costs imply the necessary inequalities

$$\Pi + A + C + X \leq 29, \quad \Pi \geq 15 + 2m, \quad \Pi \geq 15 + \lceil 3m/2 \rceil + d + z, \tag{S8.1}$$

$$A \geq \max(0, 3 - d - z), \quad 2A + C \geq 6 - m, \quad C = 0 \text{ or } C \geq 2. \tag{S8.2}$$

For $z > 0$, also $C \geq \max(2, \lceil z/2 \rceil)$; a sole Z costs at least six loops. H32 has already excluded $R < 29$ in the matching branch, so the finite routing table

uses the exact slice $\Pi + A + C + X = 29$. Its records are the integer tuples (m, d, z, Π, A, C, X) satisfying the displayed inequalities and the following additional necessary conditions:

$$1 \le m \le 6, \quad d + z \ge 1, \quad 0 \le d \le 3\binom{m}{2}, \quad 0 \le z \le 15m, \quad \Pi \le 28,$$

$$z = 0 \Longrightarrow A \ge 1; \qquad m = 1 \Longrightarrow d = 0,\ A \ge z + 1,\ C \ge z,$$

$$m = 1, z = 2 \Longrightarrow \Pi \ge 21; \qquad m = 1, z \ge 3 \Longrightarrow \Pi \ge 14 + 3z.$$

For $z = 0$, $A = 0$ would require three unit matching D and $\Pi \ge 30$. For $m = 1$, distinct helper classes and their arrays give the displayed costs; $z \ge 3$ then implies $R \ge 15 + 5z \ge 30$. These rules define exactly 1,559 records, of which 678 have $Z = 0$ and are handled in S9. Zero helpers are separate and permit every attachment count. The records describe necessary costs, not supports or feasible representations.

Explicit preliminary routing. The following table applies its rules in order to the exact envelope. Every removed record has the indicated proof; the remaining records are passed to the detailed Z cases in S10. Counts refer to integer records, and a geometric exclusion may remove one placement of a record while other placements remain.

Records	First applicable rule	Proof
678	$z = 0$: route to the doubled-helper cases	S9
123	$z > 0$, $m \ge 4$	S15, high-attachment cases
182	$C < z$: incompatible with the chosen normal form	(S4.11)
37	$m = 2$ or 3, $z = 1$	Lemmas SL9, SL10
164	$m = 2$, $d \ge 2$ or $d + z \ge 4$	Lemma SL11
25	$m = 2, d = 0, z = 2, A = 1$: requires $\Pi \ge 23, C \ge 4$	Array costs below
18	$m = 2, d = 0, z = 2, A = 2$: requires $\Pi \ge 22, C \ge 4$	Array costs below
4	$m = 2, d = 0, z = 2, A = 3$: requires $C \ge 4$ or $\Pi \ge 22$	Array costs below
116	$m = 2, d + z = 3$: requires $\Pi \ge 24, A \ge 2$	Class costs below
212	Remaining Z records	S10

Here are the costs behind the last four cuts. With two Z and no D, $A = 1$ requires two distinct canceled matching arrays, hence two $Z = 2$, $\Pi \ge 23$, and $C \ge 4$ by the unpaired matching projection. At $A = 2$, both helper blocks must be in M and at least one must cancel, giving $\Pi \ge 22, C \ge 4$. At $A = 3$, either both blocks lie in M, giving $C \ge 4$, or one lies outside and the matching one must cancel, giving $\Pi \ge 22$. For three helpers on two attachments, the sole-pair-in-one-block configuration is excluded by Lemma SL12. Every remaining placement has a multiclass endpoint, so

$\Pi \geq \lceil (13 + 8 + 20)/2 \rceil + 3 = 24$. If $A = 1$, two matching arrays must cancel at the two separate active endpoints; each needs a $Z = 2$. The third helper lies outside those blocks and makes an additional nonzero array. Thus $A \geq 2$.

The remainder is exactly

Family	Records	Additional conditions
$m = 1$, Z present	40	One or two helpers
$m = 2, d = 0, z = 2$	37	Distinct blocks; all physical attachments retained
$m = 2, d = 0, z = 3$	1	$(\Pi, A, C, X) = (24, 2, 3, 0)$
$m = 2, d = 1, z = 2$	4	$(24, 2, 2, 1)$, $(24, 2, 3, 0)$, $(24, 3, 2, 0)$, $(25, 2, 2, 0)$
$m = 3$, Z present	130	$z \geq 2$, $C \geq z$

The same cost tuple can occur with different (m, d, z); its helper counts are part of the record. The independent finite-control checker (Section S12.8) assigns all 1,559 records to these rules and retains all 212 residual records for inspection.

S8.1. **Zero flow: the signed private bound P27.**

Lemma SL7. *An effective N-free private layer with center masses one and $H^0 = 2M$ has support $P_0 \geq 27$.*

Proof. S6 gives $P_0 \geq 26$. At $P_0 = 26$, total line cost is 52 and there are nine or ten nonzero-marginal lines. Nine again fail the reciprocal -1 argument. The distribution $(3, 2, 2, 1, 1, 1)$ fails for the same reason. The only remaining distribution has four doubled centers and two simple centers, with nine F5 lines and one F7, and no nonempty zero line.

Let the simple centers be s, t, their mates p, r, and the remaining matching edge qv. The masses at p, r are $(a, 1 - a), (b, 1 - b)$ with $a, b \in \{3, 4\}$. Only q, v can supply reciprocal -1 to a simple center. Suppose only q does. Its negative half is $\{q, s, t\}$ and has mass -1. In this split the rows at s, t are empty because their sole halves contain their matching mates. Each opposite column therefore has at most its row-q glue. It cannot be the exceptional F7, which needs three glues. The negative q line, whether sparse F5, dense F5, or F7, has a primary glue $1/2$, forcing an opposite line of mass -1. Neither p nor r has such a line, so v must also have masses $2, -1$. This is the required partner argument; an isolated F7 assertion would not establish it.

The matching edge qv forces the two positive halves to contain the mate and the negative halves to omit it. If $a \neq b$, absolute mass four occurs once. A unique magnitude cannot be paired with one F7 and all other lines F5: an F7 needs a partner for its primary glue, while its two auxiliaries of equal different magnitude need two distinct partners. Thus $a = b \in \{3, 4\}$.

The split containing the F7 has either four or six nonempty lines. If one opposite F5 is dense, it forces six. The four remaining F5 lines at p, r, q, v must then be two sparse glue pairs, since their signed masses cannot form a dense four-line star. The qv equation puts the selected q, v lines in the same

half; the pr zero equation puts p, r in opposite halves. Consequently s, t lie in the same half, giving $H^0_{st} = 1$ instead of zero.

In the four-line case all three opposite F5 lines are sparse. Their masses are $c, d, -d$, where c is the F7 mass and $|c| \neq |d|$. The forced mass list has only the following possibilities. A simple F7 can occur only at $a = 3$, with a simple primary partner and auxiliaries $+2, -2$; its mate equation fails. A primary qv pair lacks the required distinct opposite-sign auxiliary centers. A primary pr pair has auxiliaries $+1$ at a simple center and -1 at q or v. Its mate equations either fail directly or force $a = c = 3$. In that last case, the remaining F5 lines contain one mass $+1$ and one mass -1, with no other absolute-mass-one partner. They cannot be dense and cannot be a sparse pair, because sparse partners have equal signed mass. This is the last contradiction.

The archived 80 six-line and 40 four-line incidence controls check the role/half bookkeeping in these alternatives. The argument permits the free rational auxiliary parameter of F7 throughout. □

For zero helpers and zero flow, $P_0 \leq \Pi$ and $A \geq 3$. The lemma therefore gives $R \geq 27 + 3 = 30$ at every attachment count.

S8.2. **Four-flow-block boundaries.** Suppose flow is nonzero. Then $C \geq 6$, leaving $P_0 \leq 20$. The corners with four flowing blocks and $C \geq 8$ reduce to

$$(m, \Pi, A, C) = (0, 15, 6, 8),\ (1, 17, 4, 8),\ (0, 18, 3, 8),\ (1, 18, 3, 8).$$

At private fifteen there are 60 dense and 120 sparse six-block cores. In the matching case the six alternating columns would have to occupy the three M arrays with capacity two. The actual incidence graph has a capacity-two Hall deficit for 176 cores. The other four have nine nonzero pair-node demands in $H^0 - 2M$; four flow edges can touch at most eight.

At $(m, \Pi) = (1, 17)$, tight balanced transport moves one positive effective star to a new line and removes at most one old demanded block. At least five remain, contradicting $A = 4$. At $(1, 18)$ with effective private fifteen, two N affect at most one effective block, and four N force one glue and one positive actual star. At least four demands remain.

For effective private eighteen, total line cost 36 gives four spectra: three F7; F9 plus F7; two F8; or one additional zero F6. The first two cannot pair the F7's two exceptional glue magnitudes. Two F8 force a single split, up to rows and columns, with matrix

$$\begin{pmatrix} -1 & 1/2 & 0 \\ 1/2 & -1/2 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix}.$$

Its twelve stars have distinct block owners. A possible zero-sum pair of stars is on one line, whereas a balanced h pair uses distinct lines; thus no erasure

fits. For the extra F6, the first split is

$$\begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix},$$

with one further sparse glue joining its zero-row center and empty-column center in another split. The five compatible partner splits have $13, 13, 13, 13, 10$ demanded blocks. At most two are erased by balanced N, still exceeding $A = 3$. This retains nonempty zero-marginal lines explicitly.

S8.3. **Three-flow projection and private support twenty.** Three distinct flow blocks with nonzero coefficients satisfy $\sum_i \delta_i(\mathbf{1}_{W_i} - \mathbf{1}_{V_i}) = 0$. A coordinate in precisely two source pairs would occur in only one nonzero vector, impossible. Equal sources give three independent side differences; a common-center source star would require a sum of three equal-magnitude signed values to vanish. Thus the sources form a perfect matching. All three flow magnitudes agree. The side partitions must all be straight or all crossed. The straight case has zero divergence, forcing $H^0 = 2M$ at private support at most twenty, impossible. The crossed case has

$$H^0 = 2M + t(\mathbf{1}_P - \mathbf{1}_Q), \qquad t > 0, \tag{S8.3}$$

for disjoint perfect matchings P, Q. The 27 side choices give one straight and four crossed kernels; coordinate relabeling covers all sources.

If $Q = M$, simple-simple edges in both matchings force $t = 1$, so H^0 is a unit cycle. The private-15 erasure inequality of S7 gives $\Pi + A \geq 27$, incompatible with $C \geq 6$. At private twenty, equality permits the same 47 actual forms as HE6. Their demanded-block counts have minimum six, whereas only $A = 3$ fits.

If $Q \neq M$, three negative offmatching edges would need three nonsimple centers, while at most two fit. Hence Q shares one edge with M. Simple-simple edges force $t = 1$, and the two nonsimple centers are doubled. Their four lines are sparse and glue only to one another; otherwise an F5 partner at a simple center would force masses $2, -1$, too small to supply the required negative-edge reciprocals. The doubled masses are $a, 1 - a$ with $a = 3$ or 4. The halves and remaining simple glues are then forced.

In the archive's zero-based coordinates, the two private templates for $M = 01 \mid 23 \mid 45$ have glues

Half	Endpoints	$a = 3$	$a = 4$
023	$2, 4$	$-3/2$	-2
025	$2, 4$	1	$3/2$
012	$0, 5$ at $a = 3$; $0, 3$ at $a = 4$	$-1/2$	$-1/2$
014	$1, 3$ at $a = 3$; $1, 5$ at $a = 4$	$-1/2$	$-1/2$

Each has four glues and sixteen stars. At a doubled center the star values are $a/2, a/2, (1-a)/2, (1-a)/2$, with no nonempty zero-sum subset for $a = 3, 4$.

No balanced erasure is possible. Their sixteen stars have sixteen distinct block owners, contradicting $A = 3$. These compatible private/flow templates are not full MAX6 identities; their existence is why private-nineteen positivity cannot be extrapolated to private twenty. The zero-helper branch is now exhausted.

S9. Matching corrections with doubled helpers

Assume $Z = 0$ and at least one D. The source graph is initially a multigraph: distinct D columns may have the same source pair. Every active center, including one with only balanced N, is counted in m.

S9.1. **Initial cuts, repeated sources, and high attachments.** A single unit D on two attachments requires $\Pi \geq 22$, even with flow. If $\Pi \leq 21$, endpoint costs are $9 + 9$ or $9 + 11$. A positive glue from a cost-nine dense line forces an inactive dense F5 partner and then the common split. In the $9+9$ case its row and column marginal sums are $1/2$ and $-3/2$. In the $9+11$ case the second endpoint would need a line of marginal $-3/2$, incompatible with its cost-eleven unit-class forms. Both are impossible.

For a single D with nonzero flow at $R \leq 29$, $C \geq 6$ and the array counts leave nine coarse corners. An outside-group D forces $(\Pi, A, C, X) = (19, 4, 6, 0)$ and a half-class with zero endpoint B rows. A nonunit D in M has $\Pi \leq 20$ and is again forced to be a half-class. The three-flow demand argument leaves a positive offmatching perfect matching, but at an active endpoint the available edge value is at most $1/2$ instead of one (or the source's demand one is below the required $3/2$ in the crossing-source case). A unit D in M leaves $\Pi \leq 21$, contradicted by the preceding private lemma. Thus the surviving single-D cases have zero flow; idle loops are not removed until their budget excludes them.

With two same-source D, a nonzero separator costs at least fifteen at its endpoint: four main N require two shared together lines of cost at least five each, plus an F5 separator. Extra N only raise the bound. At private twenty-five both endpoints cost thirteen, so source balance $NT = 1$ fails. At private twenty-six, $A = 3, C = X = 0$ and the M class must be one. Costs fifteen and thirteen are forced. The cost-thirteen endpoint contradicts Lemma SL4, which requires class absolute values $1/2$.

Three or more same-source D give $R \geq 13 + 6d \geq 31$. At six attachments $\Pi \geq 27$. Repeated D incidence raises this to at least thirty. With disjoint sources, private twenty-seven permits only half-class helpers, leaving all three matching arrays nonzero. Private twenty-eight would need two cancellations, hence two unit classes and $\Pi \geq 29$. Canceling all three needs three unit D and $\Pi \geq 30$. The five-attachment and intersecting-source cuts use the same degree charges and signed star-supply bounds; the separate nonzero-helper packet in S11 retains the repeated, path, and disjoint-source subdivisions.

S9.2. **The eight single-D zero-flow cells.** After the initial cuts, the following table is the complete residual list. Here "in M" refers to the helper's group, and a is its coefficient.

m	Location/value	(Π, A, C, X)	Exclusion
2	Outside M, any $a \neq 0$	$(25, 4, 0, 0)$	R-SD-L10
2	In M, $a \neq 1$	$(25, 3, 0, 1)$	SDX1
2	In M, $a = 1$	$(24, 5, 0, 0)$	SD2A
2	In M, $a = 1$	$(25, 4, 0, 0)$	SD2A
2	In M, $a = 1$	$(26, 2, 0, 1)$	SDX1
3	Outside M, any $a \neq 0$	$(25, 4, 0, 0)$	Signed B10/source bound
3	In M, $a \neq 1$	$(26, 3, 0, 0)$	SD3M
3	In M, $a = 1$	$(27, 2, 0, 0)$	SD3M

SDX1. Allow all 270 private columns, all 120 N at the source endpoints, all twelve group-M alternatings, the unrestricted D, and any one of the 180 mixed columns. Sixteen symmetry images of the original D21 row identity annihilate 172 possible mixed additions. The remaining eight IDs are $173, 196, 412, 435, 659, 679, 968, 988$; each has a separate exact dual for its full 404-column space. Thus every possible single mixed term is included.
SD3M.. Add all N at one further center to the private columns, group-M alternatings, and unrestricted D. Each of the four choices gives a 463-column affine space and a row separator. In the two relevant cells, the nonzero M arrays consume the entire alternating budget, so no outside zero-total array, which needs at least two columns, can occur.
SD2A.. Set $D(M, U) = 1$. Add all four columns of any one outside source array meeting the demanded U block, along with the 403-column core. There are four such arrays, and all four 407-column spaces have exact duals. The local source bound makes the inactive centers cheap, so the U block has zero actual-star sum and needs alternating sum two. Group-M alternatings do not belong to this block. The remaining budget allows at most one outside zero-total array, establishing containment in these spaces.
R-SD-L10. An inactive cost-ten center need not be light: two F5 lines of marginals $-1, 1/2$ can give directed values $-1, 0, 1, 2$. The outside-group $m = 2$ proof therefore uses no such assertion. Its three nonzero M arrays and one nonzero helper array consume $A = 4$, so all alternatings lie in those four source arrays. Admit the full 407 columns. Matching-preserving relabeling leaves exactly three zero-based geometries:

$$(U, G) = (01, 01|24|35), \quad (02, 02|13|45), \quad (02, 02|14|35).$$

Each has a named-original-row dual giving $0 = 1$. This includes arbitrary balanced N and every rational D coefficient.

For the $m = 3$ outside case, the endpoint source bound is at least 23; the third active center costs at most ten and all inactive centers cost five. Its directed entries lie in $[-1, 2]$. The matching edge disjoint from the source

must be a D side. According to which side contains the third active center, its equation requires $a \geq 1/2$ while the other requires $a \leq 0$, or requires $a \geq 1$ while the other requires $a \leq 1/2$. Both are impossible.

S9.3. **Paths and disjoint sources.** The source-degree bounds leave a two-source path at three attachments or two disjoint sources at four. At a path's shared center the two-class lemmas govern the available signs and separators; the unshared endpoints have the one-class costs. The source equations and unchanged arrays leave only the low private boundaries recorded in R29_HELPER_M34_HIGH. For disjoint sources, the separating-sum equations first force their coefficients into $\{-1, 1\}$ when the separating marginals vanish. A negative unit costs at least ten at each endpoint, whereas a positive unit costs nine. At private support at most twenty-five all four endpoints must therefore be positive units. Higher private boundaries are excluded by the G11 glue bound and signed F5/F7 star-supply arguments of S5. Their explicit costs and source incidences, including balanced attachments, are retained in that component's indexed proof body.

S9.4. **The private-twenty-five boundary, with literal substitutions.** The local derivation in Section S15.3, using the endpoint equalities proved in Section S14.5, forces the common dense split $125 \mid 346$ and the matrix

$$L = \tfrac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

The four active centers each have one fixed dense N and two possibilities on their glue-free main line. This gives exactly sixteen cores of size $9G + 6S + 8N + 2D = 25$. Twelve have five block demands, two have six, and two have four. Before deleting any variable, $R \leq 29$ forces $A = 4, C = X = 0$. Thus only the two four-demand cores need literal equations.

All IDs and keys in the next two tables are zero-based archive labels. For the first core the four variable-choice N IDs are $1419, 1386, 1185, 1110$; the charged blocks are $0, 1, 9, 10$. Their complete allowed alternating set is

$$\{81, 154, 318, 387, 698, 708, 753, 923,$$
$$950, 1012, 1077, 1097, 1106, 1224, 1306, 1317\}.$$

Literal row	Equation after substituting the private core
$T(2;13;0;45)$	$A_{1077} = 0$
$g(45;01;23)$	$A_{1077} + A_{1106} = 1$
$T(2;03;1;45)$	$A_{698} + A_{1106} = 0$
$T(0;12;3;45)$	$A_{698} + A_{708} = 0$
$T(1;03;2;45)$	$A_{708} + A_{753} = 0$
$T(1;23;0;45)$	$A_{1097} = -1$
$T(0;23;1;45)$	$A_{1224} = -1$

The six distinct IDs $1106, 698, 708, 753, 1097, 1224$ therefore have values $1, -1, 1, -1, -1, -1$.

For the second core the four N choices are $1431, 1377, 1080, 1209$ and the blocks are $0, 1, 18, 19$. The complete allowed alternating set is

$$\{38, 154, 276, 387, 607, 633, 708, 773,$$
$$866, 1012, 1077, 1106, 1125, 1202, 1306, 1317\}.$$

Literal row	Equation after substituting the private core
$T(2;03;1;45)$	$A_{1106} = 0$
$g(45;01;23)$	$A_{1077} + A_{1106} = 1$
$T(2;13;0;45)$	$A_{773} + A_{1077} = 0$
$T(1;02;3;45)$	$A_{773} + A_{1012} = 0$
$T(0;13;2;45)$	$A_{866} + A_{1012} = 0$
$T(1;23;0;45)$	$A_{1125} = -1$
$T(0;23;1;45)$	$A_{1202} = -1$

Now $1077, 773, 1012, 866, 1125, 1202$ have the same six nonzero values. Both cores require at least six alternatings, contradicting $A = 4$. The source-array equations are reconstructed separately from the four block-owner restrictions. This argument is valid only after the total $R \leq 29$ budget removes loops and mixed terms.

S10. Matching corrections with physical Z helpers

After the preceding cuts, the remaining Z cases have one, two, or three total N attachments. All use the actual diagonal vertex of each raw Z direction and its stored coefficient. This section keeps the geometric partition separate from the finite affine contradictions.

S10.1. **One attachment: all forty necessary cost records.** For matching $R \leq 29$ with one active center and nonzero helpers, normalization gives $D = 0$. The complete proof is Section S15.7; the table below gives the route through it. At most one Z block lies in M, so $A \geq z + 1$. For $z \geq 3$, two N per class and an actual star give $\Pi \geq 14 + 3z$; $C \geq z$ then yields $R \geq 15 + 5z \geq 30$. Only one or two Z remain. The six location bounds and all loop/mixed allocations are listed once, in S15.7, item 3. They cover all forty necessary cost records; they do not restrict rational coefficients to a prescribed grid.

Branch	Decisive obstruction and complete proof
One Z, $C = 6$	SZ6 gives zero divergence at the source. The required separating marginal or inactive directed entry exceeds its cost budget. Item 4.
One Z, $C = 7$	At the remaining boundary, six inactive completions demand seven or nine positive blocks. A negative active star and the helper erase at most two, leaving $A \geq 5$ against budget three. This argument does not use SZ6. Item 4.
Two Z, $C \leq 3$	Projection gives opposite values. Equal prefixes need four loops; distinct prefixes leave an inactive matching edge requiring $H^0 \geq 2$ although $H^0 \leq 1$. The private-24 cell is included using D13/14. Item 5.
Two Z, $C \geq 4$	T13 removes the value-two cost-thirteen cases. Only $(\Pi, A, C, X) = (21, 4, 4, 0), (22, 3, 4, 0)$ remain. Item 6.

At the last two boundaries the equations force, respectively, 15 local forms (incidence counts $8, 6, 1$) with 7,568 inactive completions, and six local forms with 7,136 completions. Their alternating demands are at least five and eight, exceeding budgets four and three. These are censuses of forced equality cores, with faithful nonzero masks and unrestricted rational roots. The detailed proof retains the negative unit class at cost ten, balanced N, and every actual-star position. In particular, the six-loop source lemma is never extended to seven loops.

S10.2. **Three attachments: the scoped B11 argument.** Section S15.8 gives the complete three-attachment proof. Its 130 necessary records reduce by cancellation and class costs to $(d, z) = (0, 2), (0, 3), (1, 2)$. Two or more D, four or more Z, and $(1, 3)$ are excluded in item 3.

The directed-row bound used at cost eleven has the following exact scope: a helper-free active center has $f \leq 11$, the other two active centers are glue-free, and all three inactive centers are F5. Then $B \leq 2$. The proof in item 2.1 uses signed half-glues, balanced N and the actual star of coefficient one; its final parity obstruction is independently checked on 3,240 faithful masks with unrestricted rational roots. No bound for a general active cost-eleven center is inferred.

Remaining helpers	Decisive obstruction and complete proof
Three Z, no D	Cheap half or unit classes cancel too few arrays, or confine flow to M with an unpaired endpoint charge. Item 4.
One D, two Z	At private cost 25, a multiclass endpoint has cost thirteen and the others cost eight; T13 forbids value-two cancellation. The one-class alternative fails projection or the source/side/absent directed-entry test. Item 5.
Two Z, no D	Split by whether both blocks are outside M, both are inside, or one is in each. Matching-summed content or the source/side/absent test gives the contradiction. Item 6.

The half-Z aggregate used in the last row is proved in item 2.2. With two cost-eight half-Z endpoints, a helper-free cost-nine active center and three inactive F5 centers, $\sum_{a\in M} H_a^0 \le 2$. Outside blocks give a total at most three instead of six. For two inside blocks at $C = 4$, the required outside flow has magnitude $1/4$ or $1/2$, giving a total at most $2+1+1+1/2 = 9/2$. Among 504 physical projection systems, 468 are inconsistent and the other 36 give that flow bound. The mixed-location case uses B11 only under its stated hypotheses. Balanced N and mixed columns remain allowed throughout.

S10.3. **Two attachments: same attachment and same block.** After the explicit S8 cuts there are 37 two-Z records and five three-helper records. Same physical attachment does not mean the same source or block. If both Z attach at one vertex, the other N center is helper-free. The cost reduction leaves only $(23, 4, 2, 0)$ and $(24, 3, 2, 0)$, one helper block in M, one outside, and all alternatings in four source arrays. The same-block endpoint-pair normal-form reduction is applied separately; it is not identified with the same-attachment case.

For the literal same-attachment check fix zero-based attachment 0 and matching 01|23|45. The matching Z is column 1527, and fourteen other physical Z columns occur at 0. Each has four possible sole L loops from its literal S row, with coefficient minus its Z value. For every one of $14{\cdot}4{\cdot}4 = 224$ choices, substitute those two loop equations into the six physical projection rows:

$$[P_u(1527) - P_u(L)]a + [P_u(Z) - P_u(L')]b = 0.$$

The $u = 0$ equation is $a + b = 0$. Of the choices, 208 give an explicit combination forcing $a = 0$; sixteen survive, in four outside blocks. For each block and each of the five possible second N centers, admit

$$270 \text{ private} + 120N + 2Z + 16 \text{ source-array } A + 4L = 412$$

columns. All compatible loop pairs are contained. All twenty reconstructed spaces have a saved exact row separator. The other forty-one arrays are zero identically on each containing space. Thus the filter plus the duals cover all unfiltered placements, with no sign convention inferred from an ambiguous row-key endpoint.

S10.4. **One block in the matching and one outside.** Separate zero from nonzero flow in every Z-free block. Idle loops remain allowed in both branches.

No third flow. The complete argument, including the four physical source cases, is in Section S15.6. The two-block projection has unrestricted variables $(a, b, \delta_1, \delta_2)$. Equal sources force both flows zero; different sources force $|a| = |b|$ and $|\delta_1| = |\delta_2| = |a|/2$. At $\Pi \le 22$, active costs are at most twelve and inactive costs at most nine. At $\Pi = 23, A = 3$, cancellation forces magnitude two and costs $11, 11, 5, 5, 5, 5$. The physical source/side cases then

contradict the flow equations and the inactive binary rows without assuming a generic active $B \le 2$ bound. The only remaining corners are

$$(\Pi, A, C, X) = (23, 4, 2, 0), (24, 3, 2, 0).$$

Their two loops are in the two helper blocks; all alternatings belong to the three M source arrays and the outside helper array. An extra zero-total array would need two more columns. The corrected 426-column spaces admit all private and allowed N, those source arrays, both helpers, and all loops permitted by the branch. Seventy exact separators cover all twenty-eight physical placements left by the projection reduction.

The distinction between array and block membership is concrete. In zero-based IDs the source array $(M, 01)$ contains $1289, 1291, 1295, 1298$, whereas the alternating block $(M, 01)$ contains $154, 387, 708, 1012$. The latter list cannot replace the former in a containing-space argument. Only the corrected source-array spaces are used here.

A nonzero third flow. The unit boundary is $(\Pi, A, C, X) = (21, 4, 4, 0)$ with $a = b = 1$, two active cost-nine centers, and four inactive F5 centers. Its eighty-four physical affine spaces split into sixty-eight inconsistent spaces and sixteen consistent ones. The latter are excluded by their costs: an active unit center has directed entry -1 toward its source mate, and an inactive F5 center has entries zero or one.

Outside Z IDs (zero-based)	Forced contradiction
$1307, 1267, 1345, 1337, 1324, 1268,$ $1330, 1325$	Outside-source $H^0 = -1$ forces reverse entry -1 at an inactive mate.
$1028, 1025, 724, 721$	$H^0_{23} = 2$ between inactive centers, whose maximum is one.
$384, 377, 151, 144$	$H^0_{45} = 2$ between inactive centers, again impossible.

The sixteen flow assignments and all 240 edge values are retained in the exact obstruction certificate. No claim that these sixteen affine spaces are inconsistent is made.

The other third-flow costs are separate branches: positive and negative half-class values at four loops, arbitrary two-loop flow at five loops, and the mixed-term boundaries. Their signed source/side demand proof is the one-in/out component indexed in S11. Its positive-X cover has sixteen necessary tuples, eleven at $R = 29$, including $(20, 6, 2, 1)$. Neither the unit classification nor the no-third-flow duals are applied to those cases. The common conclusion follows only after all these cost branches have been joined.

S10.5. **Both helper blocks in the matching.** This family illustrates why a consistent affine relaxation may still be incompatible with the support budget. Use zero-based $M = 01|23|45$ with physical Z columns 1527 at 0 and 1266 at 2, of stored coefficients a, b. Their arguments are

$$x_0 + m_{01} - m_{23} - m_{45}, \qquad m_{01} - x_2 - m_{23} + m_{45}.$$

The differing argument orientations do not negate b. Every group-M flow is constant on its pairs, so the antisymmetric charges force outside flow and $C \geq 4$. If exactly one outside block flows, its matching must contain 45. There are two possible matchings and three source choices, giving the six projection solutions below.

Block	Source	b/a	δ_U/a	δ_V/a	δ_{out}/a
9	02	1	1/2	1/2	−1
10	13	1	−1/2	−1/2	1
11	45	1	0	0	1/2
18	03	−1	1/2	1/2	−1
19	12	−1	−1/2	−1/2	1
20	45	−1	0	0	1/2

Division by $a \neq 0$ uses homogeneity and places no bound on its rational value. Rank witnesses and all fifteen reconstructed H^0 entries for each solution are in the symbolic certificate.

For $\Pi \geq 21$, let c count a, b equal to two. Start with $(\Pi, A, C, X) = (21 + c, 3 - c, 4, 0)$, $c = 0, 1, 2$, and allow at most one unit of extra support in any entry. These are fifteen tuples, including three at $R = 28$ and twelve at $R = 29$. There is only one outside flowing block and all alternatings lie in M. Inactive centers are light. Blocks $11, 20$ would need two loops in each helper block and two outside, exceeding the budget. Blocks $18, 19$ force $H^0_{45} = 2$ at inactive centers. Block 10 requires reverse directed entry three when no array cancels, or two cost-thirteen separators when both cancel; both fail. Block 9 with both cancellations forces inactive $H^0_{13} = 2$.

For block 9 without cancellation, the inactive edge bounds force $a = b = 1$. Both active costs are at most eleven, so every nonempty line there contains its source mate. The forced inactive halves then remove all four possible stars from block $(M, 01)$; its alternatings are outside group M and absent as well. Its exact block equation reads $0 = 1$. For example the full-column row combination has nonzero terms

$$q_{139} + q_{154} + q_{387} + q_{397} + q_{692} + q_{708} + q_{1012} + q_{1032} - q_{1392} - 2q_{1437} - q_{1527} = 0,$$

and the stated supports reduce it to $-a = 0$.

At $\Pi = 20$ the active centers are glue-free half-classes and inactive centers are F5. With one outside flow, the table forces $H^0_{45} = 3/2$, impossible. With two outside flows, $C = 6$ and each helper block has one loop of magnitude $1/4$. Each Z contributes a net $1/2$ on its source and one side. The remaining demands on $01, 23$ are strictly negative; the demand on 45 is nonpositive. Two flows can have only two negative endpoints, so the latter demand is zero. Conservation forces exactly two positive offmatching nodes of demand one and two flows of magnitude one from the source pairs. Those positive nodes are inactive pairs. At coordinate 0, the outside projected divergence is -1

and the matching-helper contribution is $1/4$, giving $-3/4$ where the projection equation requires $-1/4$. This excludes the last case without bounding an arbitrary flow coefficient on a grid.

S10.6. **Both blocks outside and the three-helper remainder.** Two outside blocks need $A \geq 5$, so $\Pi \leq 22$ and $C \leq 4$. With $C \leq 3$ there is no third flow and the two-block projection gives equal helper magnitudes. At least one active center costs at most ten, so the common magnitude is $1/2$ or one. Some matching edge belongs to neither outside matching; it requires $H^0 = 2$ without helper or flow. Inactive endpoints give at most one. With one active endpoint, the required directed value is three or four; the exact two-line equations exclude these at the available cost except a negative-unit F7 case. That exception needs an unusual glue magnitude, while its positive-unit partner and all inactive centers have half-magnitude glues. With two active endpoints, the cheap signed class forms cannot have directed sum four. In the two negative-unit case, their magnitude-one glues must pair across opposite halves, leaving each directed contribution toward the other at most one.

At $C = 4$, equality gives $\Pi = 20, A = 5, X = 0$ and half-class endpoints. No third flow leaves the same absent matching-edge contradiction. One third flow leaves nonpositive demands on all matching edges, at least two strictly negative, but it has only one negative divergence node. Thus all two-outside cases fail.

The three-helper remainder consists of one $(d, z) = (0, 3)$ cell,

$$(\Pi, A, C, X) = (24, 2, 3, 0),$$

and four $(1, 2)$ cells:

$$(24, 2, 2, 1),\ (24, 2, 3, 0),\ (24, 3, 2, 0),\ (25, 2, 2, 0).$$

Three Z in three blocks require two matching value-two cancellations, whose class costs exceed the active budget. In two blocks, the outside endpoint pair has zero β, while the matching singleton has value two. Projection forces the pair values to have magnitude one. A multiclass endpoint then costs at least fourteen and the other at least nine, exceeding the available sum twenty-two.

With one D and two Z, the D must share one of the helper blocks; both-in or both-out locations exceed projection or array budgets. There is no third flow and the two Z magnitudes agree. For distinct-prefix classes of unequal magnitudes at cost at most fourteen, solving (S4.1) leaves only a glue-free collision and class sum $1/2$. If D shares the matching block, this forces a nonpositive outside contribution and a weak matching side of at most $3/2$. If it shares the outside block, array counting requires a matching value two or zero outside β. All but one cost case similarly fail source/side weight two. The last has active class data $(2, -1)$ at cost fifteen and a unit class at cost nine.

Every crossing glue in that last case is a signed half. Five N would need four lines; four N reduce to a shared F7 plus two glue-free F4 lines; six N leave one glue and one actual star. In either feasible local form the multiclass center has exactly one nonzero marginal, $+1/2$. The dense unit line forces all opposite inactive lines into the same split with marginal $-1/2$. A positive crossing forces the multiclass line there too. The row marginal sum is 1 or $1/2$, while the column sum is $-3/2$, impossible. The six-N balanced case is retained in this argument.

S10.7. **The matching join and active scope.** Zero helpers, D only, and Z present partition the normalized identities. The initial cuts remove high attachments and unsupported helper counts; the remaining Z families are exactly the one-, two-, and three-attachment cases above. Their geometry splits include same attachment, same-block endpoint pairs, all locations relative to M, extra flowing blocks, idle loops, and positive mixed support. Every residual family has either an exact containing-space contradiction or a cost contradiction in a consistent space. Hence no matching identity has $R \leq 29$.

The global theorem follows by joining this conclusion to H32 and the nonmatching exclusions. The sixty-eight affine contradictions and sixteen cost contradictions in the unit-third-flow branch are separate obligations; the both-matching branch likewise includes its consistent projections. Each affine certificate is used only after containment in its allowed column set has been proved.

S11. Computational lemmas and the evidence register

The proofs use three different kinds of finite statement: reconstruction of the model, exhaustive coverage of a reduced family, and exact exclusion of its members. These statements should be checked in this order. An inconsistent matrix becomes relevant to the theorem only after its columns are shown to contain every representation left by the preceding reductions.

S11.1. **Exact predicates and their mathematical roles.** All arithmetic in a certificate check is over integers or reduced rational numbers. Numerical rank, a floating-point residual, and a reported optimizer gap are not substitutes for the following predicates.

Dictionary and chamber reconstruction.:
Generate balanced feature multisets of positive mass one or two, cancel common features, divide by their integer content, and identify opposite directions. For every chamber, reconstruct the gap normal from the actual feature vector. Group primitive normals with both positive and negative entries and require the weighted jump sum to vanish. Reconstruct the Boolean values and the feature-linear translation slope independently. This is the all-domain criterion in the main paper, not merely interpolation at sampled points.

Canonical transfer and literal row fidelity.:
The compression calculation must associate every raw column with its touching set, kink profile, and Boolean content, then give each eliminated profile its support-preserving replacement. The hosting calculation must cover every proper-set content left by that replacement. Finally, evaluate each retained direction and correction on the literal rows and compare the resulting rational matrix with the released model. Checking a stored matrix against itself would omit this semantic step. The analytic reason these calculations preserve arbitrary rational identities is S3.

Support coverage and local cost.:
Enumerate supports only after the analytic reduction has limited their shapes. The coefficients remain rational variables. For every feasible local shape, test the exact private-line, class, and marginal equations; charge every individual nonzero column. A canceled aggregate does not erase its summands. The incidence controls in S4–S5 establish the source array and block owner of each literal column. Cost-tuple enumeration is a separate necessary-condition check and is never used as a list of all representations.

Affine inconsistency.:
For a reduced instance $Ac = b$, publish the complete allowed-column list and a rational row y with

$$y^T A = 0, \qquad y^T b \neq 0.$$

The checker reconstructs A, b from the literal model and multiplies both products exactly. A zero column in the selected equations stays allowed; it cannot be silently deleted from the support accounting. The containing spaces used for the physical-Z cases are specified in S10.

Hall and literal-row obstructions.:
For a Hall witness, reconstruct the demanded blocks and their permissible source arrays, then verify a set U with $|N(U)| < |U|$. For a literal-row obstruction, substitute the complete allowed support into the stated rows and check the resulting equations exactly. The 180 HE5 cores, the matching zero-helper family, the two HE6 residuals, and the private25 triangular systems are distinct checks with distinct input families.

Consistent affine spaces with excessive support.:
When the affine system is consistent, a contradiction requires additional equations or support costs. In the unit-third-flow family, 68 of 84 systems have affine separators; the other 16 fail the stated B/H^0 cost conditions. The both-matching physical projection likewise has consistent cases. S10.5 excludes them using the active and inactive line budgets. A certificate declaring all of these spaces inconsistent would prove a different and false statement.

The retained-cap21 search is an exhaustive algorithm rather than a short dual list. Its input incidence, sound support relaxation, complete branching, pruning predicates, and termination condition are specified in S6.1 and in

the indexed source. A visited-node count is a useful execution diagnostic; it is not a mathematical certificate on its own.

S11.2. **Internal proof map.** The proof is organized by mathematical dependence rather than by the order in which the cases were discovered.

Ingredient	Complete derivation
Dictionary transfer and hosting	S3; main paper's canonical theorem.
Literal local laws and equality forms	S4–S5.
Retained support at least 22	S6.1, including the complete search rule.
Preliminary global bounds 27 through 31	Sections S13.1–S13.5.
Matching retained support 28	S6 and Section S14.
Nonmatching width 32	S7 and the main paper's correction census.
Zero helpers at retained support 29	S8.
One doubled helper and the critical private-25 case	S9.
Higher-cost doubled-helper and high-attachment cases	Sections S15.2–S15.5.
Physical-helper cases	S10; Sections S15.6–S15.8.
Global conclusion	The main paper's complete width-32 join.

All containing spaces use actual source arrays and physical attachments. The private-25 argument assumes $R \leq 29$. The unit-third-flow branch uses 68 affine contradictions and 16 additional support-cost contradictions. The both-matching branch retains its consistent projection systems and excludes them by the local budgets. These scope conditions are stated in the proofs that use them.

The source index `evidence/source-index.json` identifies 35 logical nodes and 122 primary source files by content hash. It also records the precise predicate and limitations of each finite record. The proof-integration map identifies the component derivations incorporated into Sections S13–S15. These indexes provide traceability; neither a matching hash nor an acyclic graph establishes a mathematical lemma.

S12. Reproduction, formal evidence, and release scope

The compact deposit contains the manuscript/checker release and complete Lean sources. The historical research archives and offline verification environment are retained locally and excluded from the deposit. Section S12.2 specifies the applicable commands. The direct checkers need Python 3.10 or later and only its standard library; they do not read historical success flags as evidence of mathematical truth.

S12.1. **Fresh checks of the two upper identities.** From the release directory, run

```
python3 code/check_upper.py
```

The default inputs are the native MAX5 and MAX6 JSON certificates in `evidence/`. For each term the checker requires a nonzero rational coefficient, a primitive balanced integer direction, positive mass one or two, and a projective direction not used by another term. It then checks the translation slope, all chambers, and every Boolean ray by exact rational arithmetic. Failure raises an error and returns a nonzero process status. The correspondence with the all-real identity is the chamber criterion proved in the main paper.

Input	Directions	Chambers	Interior groups	Boolean points
MAX5	6	120	16	32
MAX6	33	720	1,472	64

Both executions passed for this edition. MAX5 has three mass-one and three mass-two directions; MAX6 has three mass-one and thirty mass-two directions. Both free terms have translation slope one. The fresh receipt is `verification/upper-identity-checks-r20.json`. The construction tables in the main paper are generated directly from these same native JSON files, so the displayed coefficients have no separate manually transcribed source.

Hexagon comparison. For Myrisiotis's hexagon identity (arXiv:2610.04256v1, Definition 3.1 and equation (2)), run

```
python3 code/check_hexagon.py \
  --output new-hexagon-check
```

The script reconstructs the indexed forms, makes every direction primitive, absorbs orientation changes into the free term, and combines coefficients. The result has 39 nonparallel PM2 directions: six of positive mass one and thirty-three of positive mass two. The unchanged chamber checker verifies all 720 chambers, 1,536 interior groups, all 64 Boolean inputs, and translation slope one. The normalized identity and its provenance are in `evidence/hexagon-2026-11-02/`; the execution receipt is `verification/hexagon-checks-2026-11-02.json`. Table 6 uses the common conversion $(27, 39 + 2)$. The source's separate $(17, 41)$ implementation is given in its Corollary 4.2; that gate count is attributed to the source.

The eighteen-neuron MAX5 implementation. Write $r_{ab} = \rho(x_a - x_b)$ and order the first-layer outputs as

$$z = (r_{12}, r_{15}, r_{25}, r_{34}, r_{35}, r_{45}, r_{52}, r_{54}, \rho(x_5), \rho(-x_5))^T.$$

The first matrix has rows $e_a^T - e_b^T$ for those eight ordered pairs, then $e_5^T, -e_5^T$. Put

$$u_5 = z_9 - z_{10}, \qquad u_2 = u_5 + z_3 - z_7, \qquad u_4 = u_5 + z_6 - z_8.$$

The feature readouts are $m_{12} = u_2 + z_1$, $m_{34} = u_4 + z_4$, $m_{15} = u_5 + z_2$, $m_{25} = u_5 + z_3$, $m_{35} = u_5 + z_5$, $m_{45} = u_5 + z_6$, and $x_5 = u_5$. Substitute these linear expressions into the six hinge arguments of the main paper and into $L = -m_{15}/2 + m_{35}/2 + x_5$. The eight second-layer rows are those six arguments followed by $L, -L$; the output row is $(1/2, 1/2, -1/2, -1/2, 1, 1/2, 1, -1)$. All biases are zero. The three matrices contain 138 entries, 62 nonzero.

The exact matrices are `evidence/max5-network-10-8.json`. Run their independent all-real check from the manuscript/checker release:

```
python3 code/check_network.py --input
  evidence/max5-network-10-8.json
```

The cone argument below applies with $n = 5$ and anchor x_5: all 240 cones, 1,200 generating-ray values and 112 interior kink groups passed. The receipt is `verification/network-10-8-r20.json`. This proves a network construction bound; it does not assert unrestricted optimality or a new Lean verification of the conversion.

The fifty-one-neuron MAX6 implementation. The main paper's network corollary uses the same 33-direction identity and shares its eleven pairwise features. Here is a complete specification of the linear maps. With $r_{ab} = \rho(x_a - x_b)$, order the first-layer outputs as

$$z = (r_{12}, r_{13}, r_{14}, r_{15}, r_{23}, r_{24}, r_{25}, \\ r_{34}, r_{63}, r_{64}, r_{65}, r_{32}, r_{42}, r_{52}, \rho(x_2), \rho(-x_2))^T.$$

Thus the first weight matrix $A \in \mathbb{Z}^{16\times 6}$ has the corresponding rows $e_a^T - e_b^T$, followed by $e_2^T, -e_2^T$. Define the linear readouts

$$u_2 = z_{15} - z_{16}, \qquad u_3 = u_2 - z_5 + z_{12},$$
$$u_4 = u_2 - z_6 + z_{13}, \quad u_5 = u_2 - z_7 + z_{14}.$$

They satisfy $u_j = x_j$ for $j = 2, 3, 4, 5$. The required feature readouts are

Feature	Readout	Feature	Readout
m_{12}	$z_1 + u_2$	m_{25}	$z_7 + u_5$
m_{13}	$z_2 + u_3$	m_{34}	$z_8 + u_4$
m_{14}	$z_3 + u_4$	m_{36}	$z_9 + u_3$
m_{15}	$z_4 + u_5$	m_{46}	$z_{10} + u_4$
m_{23}	$z_5 + u_3$	m_{56}	$z_{11} + u_5$
m_{24}	$z_6 + u_4$		

Let $B \in \mathbb{Z}^{11\times 16}$ be this readout matrix, with rows in lexicographic feature order $12, 13, 14, 15, 23, 24, 25, 34, 36, 46, 56$. Write a_t for the coefficient vector of the main paper's argument g_t in these eleven features, and let ℓ have entries $-1/3$ at 14, $4/3$ at 56, and zero elsewhere. The second matrix and

output row are

$$W_2 = \begin{pmatrix} a_1^T B \\ \vdots \\ a_{33}^T B \\ \ell^T B \\ -\ell^T B \end{pmatrix} \in \mathbb{Q}^{35\times 16}, \qquad W_3 = (c_1, \dots, c_{33}, 1, -1).$$

The network is $W_3\rho(W_2\rho(Ax))$; every bias is zero. For example, the free-term readout is

$$\ell^T Bz = z_{15} - z_{16} + \frac{-z_3 + z_6 - 4z_7 + 4z_{11} - z_{13} + 4z_{14}}{3}.$$

The complete rational matrices and feature maps are stored in `evidence/max6-network-16-35.json`. Its gate labels use coordinates $0, \dots, 5$, corresponding to $x_1, \dots, x_6$ here; the matrix row order is exactly the displayed order. The three matrices have $6\cdot 16 + 16\cdot 35 + 35 = 691$ entries, of which 327 are nonzero. These are counts for the specified implementation, not runtime estimates.

Independent all-real matrix check. Run the standard-library checker directly on the exported matrices:

```
python3 code/check_network.py
```

It divides each of the 720 coordinate-order cones at $x_2 = 0$. For a permutation π and a sign $\varepsilon \in \{-1, 1\}$, the resulting closed cone is generated by

$$\varepsilon\mathbf{1}, \qquad v_k = \mathbf{1}_{S_k} - \mathbf{1}_{\{2\in S_k\}}\mathbf{1} \quad (1 \le k \le 5), \qquad S_k = \{\pi(k+1), \dots, \pi(6)\}.$$

Indeed, every point in that cone is $|x_2|\varepsilon\mathbf{1} + \sum_k (x_{\pi(k+1)} - x_{\pi(k)})v_k$. The checker verifies from A that each first-layer preactivation has one sign on the six generators, so its ReLU is linear throughout the cone. It then forms every second-layer argument on the six generators. Arguments with both positive and negative entries create interior kink hyperplanes; grouping their primitive normals modulo sign and summing their weighted jumps checks cancellation exactly. Once those jumps vanish, the network is linear on the cone, so its values on the six generators determine it. Agreement with the maximum there proves agreement throughout the cone. The closed cones cover $\mathbb{R}^6$, including all ties and negative inputs.

The execution passed all 1,440 cones and 8,640 generator checks; all 3,928 interior kink groups canceled. The receipt is `verification/network-16-35-r20.json`. This matrix check is separate from the Lean verification of the PM2 optimality theorems. The network establishes an upper bound of 51 hidden neurons; unrestricted neuron-count optimality remains open here.

S12.2. **Distributions, integrity, and reproduction.** The compact Zenodo package contains both PDFs, `paper-source.zip` and the complete Lean source compendium. The three large archives below are retained locally, not

deposited. Their filenames, sizes, hashes and verification scopes are recorded in `evidence/distributions-r24.json`. Downloading only the two PDFs supplies neither executable sources nor proof objects.

Manuscripts and direct checkers. The lightweight `paper-source.zip` contains both TeX sources and PDFs, the authored sections, direct Python checkers, small certificates, source indexes and execution records. From its extracted root, run

```
python3 code/verify_release.py
python3 code/check_consistency.py
python3 code/check_upper.py
python3 code/check_network.py
```

The first command checks the released file manifest and the 35-node, 122-source index against the historical archive manifests. It reports separately whether the large archive files are present. It does not execute a mathematical lower-bound proof or Lean. The second checks manuscript references, evidence routes and correspondence records. The last two check the displayed identities and the MAX6 network by exact arithmetic; the MAX5 matrix command is given in S12.1.

Historical computational evidence. The two historical archives below are not included in the Zenodo deposit. For a separately retained copy, place them in the extracted manuscript release's `evidence/` directory to run

```
python3 code/verify_release.py --archives
```

This additionally requires both archives and checks every decompressed member against its manifest. Without the flag, missing optional archives are reported rather than treated as missing manuscript files. Neither mode executes archived mathematical programs.

Archive	Files	Preserved contents
`pm2-source.tar.gz`	3,454	Research sources, generators, inputs, finite witnesses and computational receipts.
`max5-project.tar.gz`	22,459	MAX5 sources, native inputs, build configuration, project objects and kernel evidence.

The PM2 snapshot also contains later experiments; the source index identifies the proof inputs. A checker file or matching hash is not a new execution. The command `code/reproduce_lower.py` selects computational stages from this historical research material; its source-root and output options are documented by `--help`.

Complete Lean sources in a compact edition. The included `MAX5-MAX6-Lean-Compendium-2026-11-08.zip` is the source-only reading and sharing edition. It contains every project proof source and finite source record, the scope projection, dependency pins, guides and selected receipts. From its extracted root, run

```
python3 -I -B tools/compendium.py audit
python3 -I -B tools/compendium.py expand ../expanded
```

The destination must not exist. Expansion restores 10,143 MAX5 and 16,057 MAX6 compilation units, plus the separate MAX6 scope module. Keep the two projects separate: their `PM2` namespaces define different models. The audit verifies reconstruction and file identity, not Lean proof checking. Lean 4.32.2 and the external dependencies pinned in `DEPENDENCIES.json` must be installed separately to compile these sources. Compiler binaries, external libraries and compiled objects are not included in this compact ZIP.

Retained offline verification environment. The locally retained `MAX5-MAX6-Lean-Verification-2026-11-08.zip` is excluded from the deposit. It includes the pinned macOS arm64 toolchain, dependencies, accepted source and object files, and portable replay runner. For a separately retained copy, use its extracted root:

```
python3 -I -B verify.py audit
python3 -I -B verify.py verify --work ../proof-check
```

The work destination must not exist. The 8 November fresh-directory test blocked the original home directory and network: five theorem/scope statements passed replay from preserved objects and from freshly compiled roots, and ten source-compilation checks passed. This was not another complete project rebuild. For a complete MAX6 project-source rebuild, with the pinned external library objects still serving as inputs, use

```
python3 -I -B verify.py rebuild --project max6
  --work ../max6-full-rebuild
```

The accepted complete MAX6 rebuild remains the 31 October checkpoint in S12.5. The offline package README specifies the supported platform, resource limits, exact checks and their limitations. Historical absolute paths in receipts identify past runs; they are not required working paths for the portable commands above.

S12.3. **Fresh MAX5 kernel replay.** The two final declarations in `PM2/Max5Optimality.lean` have types

```
PM2.WStar_eq_six :  PM2.WStar = 6
PM2.width_six_optimal :  PM2.IsOptimal 6.
```

Their definitions, in `PM2/Definitions.lean` and `PM2/Minimum.lean`, fix the scope precisely. `Attainable w` means that there exists a `Circuit w` that is admissible and represents MAX_5 on every real input. `IsOptimal w` means

$$\mathrm{Attainable}(w) \ \wedge \ \forall v \in \mathbb{N}, \quad \mathrm{Attainable}(v) \Longrightarrow w \leq v.$$

The constant `WStar` is the least attainable natural number, defined by `Nat.find` using the proved width-six construction. Thus the first theorem identifies that minimum; the second explicitly packages both attainment at six and the lower bound for every admissible circuit, with no finite-search or coverage hypothesis left over.

An admissible `Circuit w` has 15 pairwise-max features, rational free and outer coefficients, and w nonzero integer directions. Each direction lies in `InD2`, the difference of two sums of two feature unit vectors, with repetitions allowed. Every outer coefficient is nonzero, and distinct directions are nonparallel. Here `Parallel` uses rational proportionality; for nonzero integer vectors this agrees with real proportionality, since any nonzero coordinate fixes the scalar as a ratio of integers. These conditions are exactly the rational PM2 directional-width convention of the main paper.

On 11 October 2026 the preserved final theorem types and proof values were replayed with Lean 4.32.2, kernel checking enabled and trust level zero. The harness compilation, executable link, and final theorem replay all completed successfully. The axiom set was exactly `propext`, `Classical.choice`, and `Quot.sound`; the report contains no proof hole, target axiom, or `native_decide`.

The formal definition `PM2.Circuit` assigns rational types to the feature-linear and outer coefficients. Its input type is $\mathrm{Fin}\,5 \to \mathbb{R}$, and `PM2.Represents` quantifies over every such real input. Thus the kernel theorem concerns rational coefficients and all-real input identities. The passage to real coefficients is the main paper's rationalization lemma, outside the saved formal theorem.

That run checked preserved closed proofs, rather than performing a fresh elaboration of all source modules. The preserved project contains 10,143 PM2 modules and its toolchain and dependency manifests. The external Mathlib revision is

`905b95818eb32af7874a58b427f50c1711a5e96c.`

The replay uses `scripts/recheck_kernel.py` and the recorded Lean and package locations. Machine-specific paths and the declared resource envelope are recorded in the artifact README and execution receipt. A bare `lake build` is not asserted to reproduce this saved-proof workflow.

S12.4. **Fresh MAX6 computational verification.** The following mathematical checks were reexecuted on 31 October 2026 using isolated output directories. The archived input files were preserved. The Python checks use integer and rational arithmetic; the compiled support search uses exact incidence and integer bounds.

Stage	Predicate checked and result
Raw model	All 22,155 projective directions regenerated; covering reduction and the 1,545-column matrix checked against literal feature/chamber data. Passed.
Proper-set hosting	Five-input profiles and every four-profile relation reconstructed independently in C++; 993,345 unordered profile pairs covered. Exact content bounds passed.

Stage	Predicate checked and result
Necessary support	1,530 directions, 3,951 row supports and 18 mandatory groups checked against exact equations.
Retained cap 21	Complete ordered-partition search; 16,008,595 nodes and zero closed supports. Terminated with status complete.
Local equations	Center, private, transport, source, degree, and Boolean-content duals multiplied exactly; all 27 full-column physical flow/projection identities passed.
Preliminary cases	Lower-27 through lower-31 checkers and physical-center reconciliation replayed, followed by all corrected retained-28 component controls. Passed.
Final nonmatching cases	Signed five-edge Hall obstructions and six-edge private-20 classification, erasure forms, and terminal literal rows replayed. Passed.
Final matching cases	Zero-helper, single-doubled-helper, path, disjoint-source, high-attachment, and physical-helper checks replayed. All specified predicates passed.
Physical containing spaces	Corrected same-attachment and source-array spaces rebuilt. The $68 + 16$ unit-third-flow split and all six both-matching projections passed their stated predicates.
Global dependency check	Final routing, input identities, and scope restrictions checked. This stage verifies composition records; it is not a replacement for the preceding mathematical checks.

The reproducible driver is `code/reproduce_lower.py`. It accepts an extracted PM2 source snapshot, compiles the finite search programs, invokes the selected exact checkers, and records each command and exit status. The supporting `code/replay_check.py` records the named checker, its source hash, the data accessed through its path interface, and emitted files. A timeout, failed assertion, or incomplete enumeration is a failed stage. The driver does not certify the analytic case coverage; that coverage is supplied by S3–S10 and the detailed derivations below.

The driver completed all 68 stages: 61 archived Python invocations, one independent finite-control check, three C++ compilations and three compiled executions. The direct check reconstructs the 1,559-record envelope and its complete 212-record Z remainder, and independently multiplies the twenty-one literal rows on the enlarged 403-column scope. The physical source-array check reconstructs all seventy 426-column sets and multiplies

their exact separators. The private-gap check enumerates the necessary six-line cost spectra; the crossing-line argument excluding those spectra is the proof of Lemma SL3, not an inferred computer theorem.

These exact computations provide a separately executable check of the finite ingredients. The completed Lean development also proves the rational optimality theorem, including the analytic reductions and exhaustive assembly, as recorded next.

S12.5. **Completed MAX6 Lean verification.** The accepted checkpoint is dated 31 October 2026. Its final declaration is

```
PM2.ContinuationMax6Optimality20261029.
  rational_PM2_MAX6_optimality :  PM2.IsOptimal6 33.
```

The definitions in `PM2/Definitions.lean` give this statement the following mathematical meaning:

(i) There is a circuit with 21 pairwise-max features and 33 hinges, rational free and outer coefficients, and nonzero, pairwise nonparallel directions in `InD2`, whose value is $\mathrm{MAX}_6(x)$ for every real input x.

(ii) For every natural number w and every admissible circuit with w hinges, equality to MAX_6 on every real input implies $33 \le w$.

Here `InD2` means a difference of two sums of two feature unit vectors, with repetitions allowed. `Parallel` uses rational proportionality; for nonzero integer vectors this is equivalent to real proportionality, since a nonzero coordinate determines the scalar as a ratio of integers. The free-term and outer coefficient fields are both rational. The input type is $\mathrm{Fin}\,6 \to \mathbb{R}$. These definitions match the rational model in the main paper, including its free feature-linear term.

The two displayed consequences were compiled and freshly replayed in the namespace

```
PM2.ContinuationMax6OriginalScopeAudit20261029
```

under the names

```
width33_all_real_witness
every_admissible_all_real_identity.
```

The theorem is closed: the actual PM2-to-canonical transfer, hosting, finite family coverage, physical local laws, and final lower-bound assembly are discharged in its dependencies.

Verification layer	Completed check
Source rebuild	All 16,057 PM2 modules in the actual dependency closure compiled from source in the frozen dependency order.
Independent kernel replay	The closed optimum and both explicit all-real projections were replayed with `kernel_do_check=true` and trust level zero.
Permitted axioms	Only `propext`, `Classical.choice`, and `Quot.sound`.
Scope and completion	All eight completion requirements passed machine and manual audits, including the original four-module, eight-statement assembly and all named finite families.
External inputs	Toolchain and dependency file identities and all eight package revisions were revalidated. External packages were not independently rebuilt in this run.

The compact index `evidence/max6-formalization.json` gives the exact theorem names, source hashes, accepted audit hashes, and original report locations. Copies of the two accepted completion reports accompany that index. Hashes identify the inspected records; the successful source-build and kernel-replay receipts are the evidence for execution. The 6 October computer-assisted adoption remains a historical checkpoint, superseded for formal status by this completed verification. Earlier draft comments in preserved proof sources do not describe the accepted checkpoint.

The Lean theorem establishes the rational optimum. The real-coefficient extension follows from the main paper's fixed-support rationalization lemma and is not an additional conclusion attributed to this saved formal root.

S12.6. **Artifact and submission scope.** The compact deposit includes the papers, direct checkers and inputs, and all Lean proof sources and finite proof data. The archive-dependent checks in S12.2 require the excluded historical or offline archives; Lean compilation requires separately installed pinned dependencies. The papers are CC BY 4.0; the author's Lean proofs and code are Apache 2.0, with third-party licenses preserved. The DOI for the accompanying paper is 10.5281/zenodo.24051030.

S12.7. **Correspondence between the written and formal proofs.** The tables below give entry points for a referee checking a particular mathematical step. Each module name denotes the file `PM2/<module>.lean` in the corresponding MAX5 or MAX6 project. MAX5 declarations lie in namespace `PM2`; MAX6 declarations listed here lie in `PM2.<module>`. A conditional interface is labeled as such: the closed final theorem, rather than the interface alone, establishes that its remaining inputs have been discharged.

The formal proof organizes some support boundaries differently from the written $27 \to 31$ chain. The map therefore identifies mathematical interfaces, not a one-to-one translation of section order. In particular, the Lean budget variable `q` counts live null triples; it is not the paper's $q = \#\{G : \Omega_G \neq 0\}$. Lean uses unscaled coefficients where this supplement often multiplies them

by three. The definitions and theorem types, rather than a shared short name, fix each scope.
MAX5.

Written proof	Formal module, declarations, and scope
S2: complete direction family	`CanonicalDictionary` `canonical_dictionary_complete`. Covers every nonzero PM2 direction up to integer scaling.
S2: the profile lemma	`ShapeInvariants` `canonical_top_by_histogram`; `canonical_zero_profile_top`; `canonical_equal_profile_top`. Canonical five-input directions; profile and top content have their literal semantic definitions.
S2: transfer to the stored system	`QuotientActualModel` `attainable_stored_quotient_solution`. Conditional on checked profile and Boolean fidelity; transfers an attainable width to a stored solution with no larger support.
S2: all small supports	`StoredSupportExclusion` `stored_solution_support_gt_five`. Conditional on reference coverage; the next entry discharges that premise.
S2: complete reference coverage	`CanonicalSavedReferenceProof` `canonical_saved_reference_coverage`. Supplies the three-, four- and five-anchor coverage premise.
S12.3: the optimum	`Max5Optimality` `WStar_eq_six`; `width_six_optimal`. Closed statements identifying the minimum and packaging attainment with the lower bound.

The transfer's fidelity premises are supplied by `quotient_profile_all_checked` in `QuotientProfileCatalog` and `quotient_boolean_all_checked` in `QuotientBooleanCatalog`.
MAX6.

Written proof	Formal module, declarations, and scope
S3: hosting and canonical transfer	`ContinuationFullTransfer` `representation_transfer`. Preserves the all-real identity in the actual 1,545-column dictionary and bounds nonzero support.
S3: compression budget	`ContinuationCompressionAssembly` `representation_compression`. Retains the saving of two directions per live null triple through the full transfer.

Written proof	Formal module, declarations, and scope
S4–S5: private line laws	`ContinuationPrivateLineCostSemantics` `row_equations`; `nonzero_glue_sum_cost`; `N_line_cost`. Literal private-row equations and support costs for actual canonical columns.
S4.2: source-array laws	`ContinuationArrayContentSemantics` `array_equation`; `omega_mass`. Common corrected array values and their total mass; formal coefficients are unscaled.
S4.3: physical flow and projection	`ContinuationPhysicalFlowSemantics` `F_zero`; `P_zero`. Full-column edge-flow and vertex-projection identities, including mixed-column cancellation.
S6–S10; S13–S15: endpoint reductions	`ContinuationEndpointFrontier` `selected`. Derives the remaining endpoint budgets under width at most 32 and the selected normal form.
S7: nonmatching endpoint cases	`ContinuationNonmatchingEndpointResidual` `Bridge` `nonmatching_endpoint`. Excludes the remaining nonmatching branch under the preceding endpoint bounds.
S8–S10; S14–S15: higher attachment cuts	`ContinuationMatchingThreeMultiZComplete` `Frontier` `matching_active_two`. The remaining matching endpoint case has at most two active centers.
S10: the two-attachment remainder	`ContinuationMatchingTwoDfreeComplete` `selected`. Its returned `Bounds` includes `matching_two_excluded`, closing the two-center endpoint branch.
S10; S15: one attachment, few loops	`ContinuationMatchingOneF14SelectedAssembly` `selected_low_loop_excluded`. Excludes the selected matching one-center branch with at most three loop columns.
S10; S15: one attachment, more loops	`ContinuationMatchingOneHighLoopAssembly` `high_loop_of_cases`. Excludes the complementary loop branch conditional on three local exclusion inputs.
S11–S12: discharge and final assembly	`ContinuationMax6Optimality20261029` `rational_PM2_MAX6_optimality`. Closed optimum: imports the analytic assembly and discharges every remaining finite-family input.

For MAX6, the last step applies `ContinuationMax6ReducedPhysicalChecks Assembly20261029.optimality_of_three_finite_families` to the proved signature, group-pressure and router families. Its inputs are supplied by `ContinuationF13SignatureKeyAll20261029.checked`, `ContinuationF13 ActualGroupPressureAll20261029.group_pressure`, and `Continuation MatchingOneF15ActiveRouterComplete.all`. The all-real witness and universal lower-bound projections are printed in Section S12.5.

The machine-readable map `evidence/lean-correspondence-r21.json` records exact declaration names, source locations, theorem headers and file hashes. Every listed module belongs to the corresponding closed root's import closure. This source correspondence check is distinct from the accepted kernel replay and source-rebuild evidence in Sections S12.3 and S12.5.

S12.8. **Implementation and execution register.** Paths beginning with `src/` or `results/` refer to the preserved PM2 snapshot; `code/` and `verification/` refer to this release. Mathematical families and predicates remain in the cited sections. Locators and hashes identify files; dated receipts record execution.

Mathematical use	Implementation or receipt
Eight-edge equality (S5.1)	`code/check_outer_geometry.py`; `verification/outer-geometry-r20.json`
Compression (S3.2)	`src/max6_two_row_compression.py`
Triple content (S3.3)	`src/check_null_triple_content.py`; canonical-row and weighted-model stages of `code/reproduce_lower.py`
Hosting (S3.4)	`src/audit_five_four_hinge.py`; `src/check_five_four_hinge.cpp`, with `src/check_small_corrections.cpp`
Literal rows (S4.2)	`results/max6_opus033_cell_laws.json`
Common array value (S4.2)	`code/check_array_example.py`; 3 November 2026 execution: `verification/array-example-r12.json`
Retained cap 21 (S6.1)	`src/max6_p4_search.cpp`, with `results/max6_intercept_support_incidence.txt`
Six-edge private20 (S7)	`src/check_e6_r26_forced_private20_20261005.py`
Finite controls (S8–S10; S13)	`code/check_finite_controls.py`; routing and enlarged D21 checks: S12.4. The latter use all 403 columns; see Lemma SL8
General-flow certificates (S13.4)	`src/check_global30_frontier_20261005.py`

S12.9. **Comparison conventions and full construction table.** This table expands the main paper's comparison of selected certificates. It does not assert minimum neuron counts for the original authors' methods. For size-two feature multisets A, B,

$$\max\Big(\sum_{e\in A} m_e, \sum_{e\in B} m_e\Big) = \sum_{e\in A} m_e + \rho\big((\chi_B - \chi_A)\cdot m\big).$$

An orbit class is not a direction count. Expand all labeled summands, retain their permutation multiplicities, divide each direction by its integer content, absorb orientation changes into the feature-linear term, combine parallel coefficients and delete zeros. Additional reductions modulo feature-linear functions are listed separately.

Explicit PM2 representation	Directional width MAX5	Directional width MAX6	Converted MAX6 network widths
Rueß et al.: expanded certificate	105	495	$(27, 497)$
Same certificate, modulo free features	85	465	$(27, 467)$
Wang–Basu: first identity	240	810	$(27, 812)$
Wang–Basu: second MAX5 identity	180	—	—
Our reduction in the Rueß dictionary	—	56	$(27, 58)$
Myrisiotis: normalized hexagon identity	—	39	$(27, 41)$
Optimal identities in this paper	**6**	**33**	$(27, 35)$

TABLE 6. Normalized directional widths and the common MAX6 network conversion $(27, W + 2)$.

The Rueß certificates are those of arXiv:2607.21651v2. The Wang–Basu rows expand MAX5 equations (6), (7) and MAX6 equation (8) of arXiv:2608.25221v1 as ordered patterns, retaining multiplicities. The width-56 row is our subsequent support reduction inside the Rueß MAX6 dictionary. The hexagon row reconstructs Definition 3.1 and equation (2) of Myrisiotis's paper; six doubled differences are made primitive. All these identities passed exact checks on every chamber. The implementation `code/check_baselines.py` reconstructs the first five rows, and `code/check_hexagon.py` reconstructs the hexagon row; Section S12.8 fixes the path conventions.

The common MAX6 conversion uses 27 first-layer units and $W+2$ second-layer units, including two for the free term. Tailored conversions give $(17, 41)$ and 58 hidden neurons for Myrisiotis's identity (his Section 4 and Corollary 4.2), and $(16, 35)$ and 51 for ours; our generic conversion uses 62. Feature sharing gives our MAX5 identity $(10, 8)$ and 18 neurons instead of the generic $(20, 8)$ and 28. These are all-real construction bounds, not unrestricted minima.

S13. The complete preliminary chain

These derivations supply the preliminary inequalities used in S6. Each component starts from the preceding bound. All normalizations preserve an actual PM2 identity; therefore a strict support decrease contradicts that preceding bound. This gives the order $27, 28, 29, 30, 31$ without using any retained-29 exclusion. The physical attachment and source-array conventions are those of S4. In particular, the fixed twenty-one-row separator is used only with $C = X = 0$.

Equation tags inside each component are local to that component. References to numbered items specify the enclosing section and, when needed, its lettered part; the links lead directly to the cited item. The scalar notation L_3 denotes the single-loop family L of S4; H^0 denotes effective star content, and "F5" denotes a line of cost five. All compact cost records denote equalities, not inequalities. Finite incidence and row assertions are the exact predicates of S11, with the executions recorded in S12.

S13.1. **The lower bound 27.**

Scope. Canonical identities; the retained bound $R \geq 22$ is available.
Obstruction. Private charges, source balance and the full single-D row identity exclude matching $R \leq 23$; item 9 handles nonmatching corrections.
Conclusion. $W \geq 27$.

1. Counts and an allowed local normalization. All retained coefficients are scaled by three. Every vertex has centered star mass one. A matching correction has unscaled coefficient $2/3$ on each of its three edges, so its scaled ordered slots have target -2.

Use the support counts (S4.8), with $L_3 = L$. Here N is nonalternating P4 support, Z zero-loop support, D doubled support, A alternating support, E double-loop support, and X mixed support. R is retained support; total width in the matching branch is $R+3$. Let m count vertices carrying at least one nonalternating direction as its attachment vertex. The proof assumes $R \leq 23$ and derives a contradiction.

For a loop block B with source $U = \{u, v\}$ and other pairs V,W, write

$$\zeta_B(u) = Z_B(u) + D_B, \zeta_B(v) = Z_B(v) + D_B,$$
$$z_B = \zeta_B(u) + \zeta_B(v)$$

The h equations say that each of the two nonalternating side sums at (B,u) is $-\zeta_B(u)$, and similarly at v. Thus nonzero ζ at an endpoint requires nonalternating directions attached there. Each such class needs at least two selected N directions, one on each side; classes are disjoint.

Use the full-matrix relation in the canonical quotient by the free feature-linear term ℓ.m:

$$D_B - Z_B(u) - Z_B(v) + E_B = 0$$

The pointwise proof of the pure relation is in S4.4. The two raw helper preactivations exchange with the doubled and double-loop preactivations according to the order of x_u, x_v. The stored quadruples have common orientation in each block (15 positive and 30 negative), and all 45 full-matrix relations are checked. Independent sign reversal changes only the free feature-linear term. We now prove the support consequence needed here.

This permits removing D_B whenever at least one endpoint ζ is zero. If, for example, $\zeta_B(v) = 0$, the coefficients have form

$$(D, Z_u, Z_v, E) = (d, \alpha - d, -d, e)$$

Replace them by $(0, \alpha, 0, e - d)$. If $d \neq 0$ and $\alpha \neq 0$ the old support is at least two and the new support at most two. If $\alpha = 0$, the old support is at least three and the new at most one. If $d = 0$ nothing changes. This normalization preserves ζ, stars, glues, N, A, corrections and m, and never increases **total** support, even if it introduces E.

After normalization:

- a nonzero D has two nonzero endpoint zetas, hence two N-active endpoints;
- Z vanishes at N-inactive vertices;
- when $m = 1, D = 0$;
- when $m = 2$, every D has the same source pair consisting of those vertices.

This is narrower than deleting D without qualification, which is not support-nonincreasing in general.

2. Private vertex charges, including their equality case. A private line l has three glue entries, sum s, and three V equations

$$S_y = l_y - s - \nu_y,$$

where ν_y is the sum of its two possible N coefficients. Define

$$F(l) = \text{support } (l) + 2 \cdot \text{stars } (l) + 2 \cdot N(l),$$
$$\pi(u) = 1/2 \sum_{\text{lines of} u} F(l) + Z(u) + D(u)/2$$

Each glue occurs on two lines; each star and N on one; each D has two source endpoints. Consequently $\sum_u \pi(u) = \Pi$ exactly. We use the previously checked facts:

- a nonzero-sum line has $F \geq 5$;
- a line containing N has $F \geq 4$;
- the two N directions on one h side have different private lines, and the four directions in both sides of a class occupy four different lines;

- with $\Gamma(u)$ the sum of its glue marginals and $\Psi(u) = \sum_B \zeta_B(u)$, one has $\Gamma(u) - \Psi(u) = -1/2$.

Every N-active vertex has selected N on at least two private lines. For nonzero ζ, the two opposite h sides force distinct lines. When all zetas vanish, a selected N belongs to a zero-sum h side and must have its cancelling partner on a different line. Thus the conclusion also holds for a cancelling class. If some ζ at u is nonzero, its line charge is at least four and its Z/D charge at least $1/2$, giving $\pi(u) \geq 9/2$. If all its zetas vanish, normalized Z/D vanish there and $\Gamma(u) = -1/2$. The two N lines cost at least four each, and a nonzero-sum line raises the total line cost to at least nine. Again $\pi(u) \geq 9/2$. An N-inactive vertex has $\pi(u) \geq 5/2$. Therefore

$$\Pi \geq 15 + 2m. \tag{1}$$

For matching corrections there is a useful equality consequence:

$$\Pi = 15 + 2m \Longrightarrow A \geq 3. \tag{2}$$

Indeed, equality fixes every vertex charge at its indicated minimum. At an active vertex with nonzero ζ, its helper charge is exactly $1/2$: there is one D and no Z, and its two N lines each have $F = 4$. A line with N and $F = 4$ has no glue entries. Indeed, the only glue-bearing integer-count candidate is (glues,stars,N) $= (2, 0, 1)$. For $l = (p, q, 0)$ with p,q nonzero, $l - s \cdot 1 = (-q, -p, -p-q)$ has at least two nonzero entries; one N coefficient cannot cancel all of them. Thus $\Gamma(u) = 0$ and $\Psi(u) = 1/2$, so its unique D coefficient is $1/2$. Every such D joins two vertices of this kind, and their source pairs form a matching because no vertex can carry a second D. There are no Z directions. Accordingly each aggregated source $\zeta \sum$ is zero or one. By the slot law below, every edge of the correction matching still has a nonzero alternating isolated sum, either -2 or -1. At least three alternating directions are required.

3. Exact group, slot, and projected-flow identities. For a perfect matching G and an edge U of G, let t_U^G be the total of its four alternating directions with isolated pair U. In the block (G,U) the paired $T/V/h$ equations give a common value

$$\Omega_G = t_U^G - z_{G,U}, \text{independent of} U \in G. \tag{3}$$

The checker supplies exact combinations of at most 18 local equations for the two equalities in each of all 15 groups. In particular, writing I_e for the total of all alternating directions isolated at e,

$$I_e = -2 \cdot \text{indicator } (M)_e + \sum_{B:\text{source } = e} z_B,$$

$$\sum_{G\text{contains } e} \Omega_G = -2 \cdot \text{indicator } (M)_e. (4)$$

Each alternating direction has exactly one isolated pair, so $A \geq \text{support } (I)$. Since the right side in the second equation is nonzero, at least one Ω_G is nonzero.

Let L denote L_3 support alone. Equations (3) and the S rows imply

$$2A + L \geq 6 - m. (5)$$

To see this, fix a group with $\Omega_G \neq 0$. For each of its three edges U, if $t_U^G \neq 0$, its array needs an alternating column, which can account for at most two N-inactive endpoints. If $t_U^G = 0$, then $z_{G,U} \neq 0$, so at least one endpoint is N-active. If the other is inactive, its ζ is zero and the difference of the two S rows forces an L_3 direction in that block: the Z coefficients differ, while D and E cancel in the difference. The three blocks and the three alternating arrays are disjoint. Counting the $6 - m$ inactive endpoints proves (5).

There is also a coordinate projection of the general flow law. Orient a block's flow edge $V \to W$ and let its projected column be $c_B = 1_W - 1_V$. For matching corrections,

$$2 \sum_B \delta_B c_B$$
$$= \sum_{B:\text{source} = \{u,v\}} (Z_B(v) - Z_B(u)) \cdot (e_u - e_v). (6)$$

For an explicit derivation, put

$$\beta_B = z_B, \qquad \text{Sbeta } = \sum_B \beta_B,$$
$$\text{Ubeta } (w) = \sum_{B:w \in \text{source } (B)} \beta_B,$$

and $\Psi(w) = \sum_B \zeta_B(w)$. Literal V substitution and h elimination give

$$\deg (H)_w = 2 + \text{Sbeta } /2 - \Psi(w) - \text{Ubeta } (w),$$
$$\deg (\nu)_w = \text{Sbeta } /2 - 3 \cdot \text{Ubeta } (w)/2$$

Subtracting the matching degree 2 gives Ubeta $(w)/2 - \Psi(w)$, which is half the sum of partner ζ minus own ζ over source pairs at w. The D terms cancel. This proves (6), with the factor two as written. The checker independently derives all six coordinate identities. Each c_B vanishes at its two source vertices. If $Z = 0$, the projected flow is zero, and the 45 distinct nonparallel c_B imply $b = 0$ or $b \geq 3$, where b counts nonzero δ_B. With $Z = 0$ every nonzero flow block needs at least two L_3/E directions, so $C \geq 2b$. In general only a block without Z enjoys that additive two-column charge; Z has already been counted in Π.

A block with Z present needs at least one L_3/E direction by its S rows. If exactly one endpoint Z is nonzero, it needs an L_3 direction. If there is exactly one Z direction in the entire identity, then

$$C \geq 3.(7)$$

One loop term is needed in its block. Equation (6) has a nonzero entry at that Z's source vertex, whereas that block's c_B is zero there. Some other flow block is therefore nonzero; that other block has no Z and needs at least two L_3/E directions. These costs are disjoint.

4. All ζ-zero configurations and $m \geq 3$ are excluded. When $N = 0, h$ forces every endpoint ζ to vanish, and the allowed normalization removes D and Z. Then $\Pi = P$, effective and actual stars coincide, $P \geq 15$ and $A \geq 3$. If $H = 2M$, Section S13.1, Appendix A gives $P \geq 24$. Otherwise flow divergence is nonzero. Its projection is zero by (6), so $b \geq 3$ and $C \geq 6$, giving $R \geq 15+3+6 = 24$. This closes $N = 0$ directly, without the stronger width-30 or two-partition theorems.

For the rest of this section assume $N > 0$. First suppose every ζ vanishes. Normalization removes all D and Z. Define effective stars by the private formulas $S^0 = l-s$. The actual stars are $S = S^0-\nu$, so their effective private support P_0 is at most stars + glues $+N$, which is $\Pi \leq 23$. Each N has a unique V row; an effective-star position absent from the actual support requires an N in that row, so different new positions require different N directions. The h equations give N mass zero at each center, so S^0 has center mass one. They also show H(actual stars) $= H(S^0)$: the two N directions on each h side have the same pendant pair and opposite total. The nonalternating flow source vanishes. By Section S13.1, Appendix A, $H(S^0) = 2M$ would require $P_0 \geq 24$. Hence flow divergence is nonzero; $b \geq 3, C \geq 6$, and (4) gives $A \geq 3$. Since $N > 0$ implies $\Pi \geq 17, R \geq 17 + 6 + 3 = 26$, contradicting $R \leq 23$.

Now suppose $m \geq 3$. If $m \geq 5, (1)$ already gives $\Pi \geq 25$. If $m = 4, \Pi \geq 23$; equality forces $A \geq 3$ by (2), while $\Pi \geq 24$ is already sufficient. If $m = 3, \Pi \geq 21$. At $\Pi = 21, (2)$ gives $A \geq 3$. At $\Pi \geq 22, (5)$ forces at least two directions among A and L, since $2A + L \geq 3$. In every case $R \geq 24$.

It remains to treat $m = 1$ or 2 with some nonzero ζ.

5. One attachment vertex. Let u be the only N-active vertex. Normalization gives $D = 0$, and all Z directions are at u. Write z for their number; $z \geq 1$. Two matching edges avoid u, so (4) gives $A \geq 2$. Every Z lies in a distinct block requiring a loop term, so $C \geq z$. Its distinct h classes require $N \geq 2z$ at u. There is also at least one star at u. The other five vertices contribute at least 25/2 to Π. Thus

$$\Pi \geq \text{ceil } (25/2 + 1 + 2z + z) = 14 + 3z$$

For $z \geq 2, R \geq 14+3z+2+z \geq 24$. Consider $z = 1$. The line charge at u is at least four, giving $\Pi \geq$ ceil $(25/2+4+1) = 18$. Equation (7) gives $C \geq 3$. If $A \geq 3$ the proof is complete. If $A = 2$, its two isolated sums must cover the two unaffected matching edges, at coefficient -2 each. Equation (4) forces the sole Z source to be the remaining matching edge, with Z coefficient 2.

Only $\Pi = 18$ could then leave $R \leq 23$. $\Gamma(u) = 3/2$ already forces its line cost to be at least nine, because cost eight has no glue. The total line cost is $2(\Pi - Z) = 34$; the other five vertices use at least 25, exhausting the budget. Center mass one also requires a star at u. Thus equality gives total private line cost $F = 9$ at u and cost-five lines at the other vertices. Both h sides of its nonzero class have $\sum -2$. With $F \leq 9$ there are exactly two N directions, one on each of two lines: three from the class would occupy three lines of cost at least four, and extra zero $-\zeta$ classes come in pairs, giving at least four N plus a star and hence $F \geq 10$.

The two lines have costs five and four. The cost-four line has no glues. For a line with one N coefficient n and $F = 5$ there are just two possibilities:

Glues	Stars	Glue-line sum
1	1	-n
3	0	$-n/2$

This follows directly from $F = g+2s+2 = 5$ and $S = l-(\sum l)\cdot 1 - n \cdot e_y$. With $n = -2, \Gamma(u)$ is therefore 2 or 1. But the center identity requires $\Gamma(u) = \Psi(u) - 1/2 = 2 - 1/2 = 3/2$. This contradiction closes $m = 1$.

6. Two attachment vertices: all but one helper case. Let $U = \{u, v\}$ be the two N-active vertices. All D directions have source U. Write d for their count, z for Z count, and $h = d + z \geq 1$. The inactive vertices contribute at least ten to Π and the two active line charges at least eight, so

$$\Pi \geq 18 + h. \tag{8}$$

At least one matching edge avoids both active vertices, hence $A \geq 1$. When all helper sources are U, only the isolated sum at U can change, so at least two matching edges are unaffected and (4) gives $A \geq 2$.

If $d \geq 2$, normalization ensures two nonzero endpoint zetas for every D. Thus $N \geq 2d$ at each active vertex, in distinct classes. Each vertex also needs at least one star. Their line charges total at least $2 + 4d$. Hence $\Pi \geq 12 + 4d + h$. For $h = 2$ this gives $\Pi \geq 22$ and $A \geq 2$; for $h \geq 3$ it gives $\Pi \geq 23$ and $A \geq 1$. These cases have $R \geq 24$.

We may assume $d \leq 1$. The remaining counts are:

Helpers	Exclusion
$h \geq 6$	$\Pi \geq 24$.
$h = 5$	$\Pi \geq 23$ and $A \geq 1$.
$h = 4$	$\Pi \geq 22, A \geq 1$, and some Z forces $C \geq 1$.
$h = 3, d = 0$	Three Z require at least two blocks: $\Pi \geq 21, A \geq 1, C \geq 2$.
$h = 3, d = 1$	If the two Z are in different blocks, use $C \geq 2$. If in one block, its source is U and all helper sources are U, so $A \geq 2$ and $C \geq 1$.

Two Z directions in one block occupy its two source endpoints. Both must be N-active, so that source is U. Each D contributes one unit of support to Π, split as one half-charge at each endpoint. A D source U and any Z source intersect in an active vertex, so they can affect at most one edge of a perfect matching. These observations justify the source and charge assertions in this case table and the next cases.

For $h = 2$ with $d = 1, z = 1, \Pi = 20$ would require both active line charges to be exactly four. They have no glue entries, so $\Gamma(u) = \Gamma(v) = 0$ and $\Psi(u) = \Psi(v) = 1/2$. The D contributes equally at both vertices, but the single nonzero Z adds at only one: impossible. Thus $\Pi \geq 21, A \geq 2($ one D source U plus one Z source can affect at most one matching edge), and $C \geq 1$, giving $R \geq 24$.

For $h = 2$ with $d = 0, z = 2, \Pi = 20$ likewise forces $\Gamma(u) = \Gamma(v) = 0$. There must be one Z at each active vertex, each of coefficient $1/2$. No matching isolated sum can then cancel its target -2, so $A \geq 3$ and $C \geq 1$. If $\Pi \geq 21$, two Z blocks give $C \geq 2$ and $A \geq 1$; a single Z block has source U and gives $C \geq 1$ and $A \geq 2$. Again $R \geq 24$.

For $h = 1$ consisting of one Z, the active vertex without ζ has line charge at least $9/2$; the other has at least four plus its Z charge one. Together with the inactive charge ten, $\Pi \geq$ ceil $(39/2) = 20$. Here $A \geq 2$ and (7) gives $C \geq 3$. This case is also excluded.

Only one D and no Z remain.

7. One doubled coefficient: the tight private cases. Let its coefficient be $a \neq 0$ and its source $U = \{u, v\}$. At both active vertices $\Psi = a$ and $\Gamma = a - 1/2$. Their only nonzero $-\zeta$ class has the two h side sums -a. We have $\Pi \geq 19$ and $A \geq 2$. Because $Z = 0$, any nonzero flow needs at least six loop directions, contradicting $R \leq 23$. Thus $b = 0$.

We use the following small-cost facts at an active vertex. Let f be the sum of its private line costs, excluding its D half-charge.

For a line without N, the possible nonzero-sum costs are $5, 7, 8, 9$; zero-sum costs are $0, 6, 9$. In particular there is no stray pure line of cost one or two. Together with the distinct h/V class ownership this justifies the small-cost classifications below.

- $f \geq 8$. If $f = 8$, all its glue entries vanish.
- If $f \leq 9$, there are exactly two N directions of coefficient -a, on the two sides of the nonzero class. The reason is the same as in item 5.
- If $f = 9$, the cost-five / cost-four classification gives Γ in $\{a, a/2\}$. Thus $\Gamma = a - 1/2$ is possible only for $a = 1$, with a three-entry glue line of pattern $(1/2, 1/2, -1/2)$.
- If $a = 1/2, \Gamma = 0$. A glue-bearing vertex has $f \geq 10$. At $f = 10$ there are still exactly two N directions: four N plus a star and a glue would cost at least eleven, and three from the nonzero class use three lines. The only possible glue-bearing form is one cost-six line and one cost-four line. The former has glue entries $+1/2, -1/2, 0, \sum$ zero; its N coefficient $-1/2$ cancels the negative entry. Two cost-five lines would instead give positive total Γ.

The literal V incidence has one further consequence: the private half of every N direction in this class contains both u and v. For a path p-a-b-w with isolated pair Y, its V row has center a and private half $\{p, a, b\}$. The class prefix pair $\{p, a\}$ is therefore in one half. The incidence is also checked for all $360N$ columns from their actual V rows and path metadata; no additional support classification is used.

Case $a = 1/2$**.** Equation (4) gives $A \geq 3$, so $\Pi \leq 20$. If both active vertices have no glues, $H_G(U) = (B_{uv} + B_{vu})/2 = 0$. The exact H substitution and nonalternating source then give $(H - \nu)_U = 1$. This cannot equal $2 \cdot \text{indicator } (M)_U$, contradicting $b = 0$.

If an active vertex has glues, its f is at least ten. The budget $\Pi \leq 20$ has only two units of line-cost excess over the baseline $f = 8, 8$ and four inactive costs five. Thus one active vertex has precisely the last glue-bearing form above, the other has no glues, and all inactive lines have their cost-five equality pattern with $\sum -1/2$. The glue-bearing line lies in a private half containing u,v. Its positive entry $+1/2$ crosses an inactive line of $\sum -1/2$ and cost five. That line must have pattern $(+1/2, -1/2, -1/2)$, so all three crossing entries are nonzero, including the entry on the other active vertex's line. This contradicts that vertex having no glues.

Case $a \neq 1/2$**.** Both active Γ values are nonzero, so $f \geq 9$ at both and $\Pi \geq 20$. $\Pi = 20$ would force $f = 9$ at both and equality on all inactive lines. The preceding classification then forces $a = 1$ and a full glue line of pattern $(+1/2, +1/2, -1/2)$ at each active vertex. Its two positive entries cross inactive cost-five lines of $\sum -1/2$, forcing those lines to have three nonzero entries. Since u and v belong to the same private half, both active glue lines and the third inactive line belong to that same private block. Its half containing u,v has marginal $\sum 1/2 + 1/2 - 1/2 = 1/2$, while the opposite

half has three inactive marginals summing to $-3/2$. A matrix cannot have different total row and column sums. Thus $\Pi \geq 21$.

Unless $a = 1$ and U belongs to M, (4) requires $A \geq 3$, already giving $R \geq 24$. The only remaining possibility is

$$a = 1, U \in M, \Pi = 21, A = 2, C = X = 0. \tag{9}$$

*8. A*21*-row exact contradiction for the final case.* Let G0 be the matching group of the sole D block. Every other group has $z_B = 0$. With only two alternating directions globally, (3) forces $\Omega_G = 0$ in every $G \neq$ G0: otherwise all three of its arrays would need an alternating direction. Equations (4) then imply G0 $= M$ and $\Omega_M = -2$. The two alternating directions are isolated at the two edges of M other than U, with coefficient -2 each.

Relabel the entire configuration once as

$M = \{01, 23, 45\}, U = \{01\}$. This simultaneously fixes the D block; no separate symmetry assumption is made on the star or glue data. In the fixed dictionary its doubled column is 1437. Permit all stars, all glues, all N directions attached at 0 or 1, and even all eight group-M alternating directions whose isolated pair is 23 or 45. All other retained types are zero by (9).

The following combination of **zero-based cell-row IDs** from the fixed 720-row cell-law file vanishes on every permitted column except 1437, where its coefficient is -1:

$$\begin{gathered} -1/2 \cdot (231 + 232 + 261 + 262) \\ +1/2 \cdot (233 + 263 + 276 + 277 + 279 + 280 + 306 + 307 + 309 + 310) \\ -1/2 \cdot (390 + 391 + 420 + 421) \\ -(413 + 443) + 509 \end{gathered}$$

These are six T rows, fourteen V rows and one g row. The result JSON records their full named keys, rational coefficients, the complete combined row, and all 398 permitted column IDs. The replay recomputes the combination directly from the literal rows and verifies that its restriction is exactly $-D_{1437} = 0$. This contradicts $a = 1$.

The certificate is stronger than needed: it permits arbitrary support and arbitrary signed coefficients on all allowed private columns, and all eight permitted alternating columns. The proof does not depend on the numerical behavior of a solver or on sampling coefficient values.

9. Coverage and global conclusion. Every normalized matching identity of retained support at most 23 has been excluded:

- $N = 0$: the zero $-\zeta$ argument in item 4 gives matching retained support at least 24.
- $N > 0$, all $\zeta = 0$: item 4.

- At least three N-active attachment vertices: S13.1, items 2–4.
- One N-active vertex: item 5.
- Two N-active vertices with at least two helpers, or a sole Z: item 6.
- Two N-active vertices with a sole D: S13.1, items 7–8.

All counts are support counts of disjoint direction types. In particular, Z is charged in Π, and the loop cost is $C = L_3 + E$; the argument adds $2b$ only when the charged flow blocks contain no Z. Mixed support is always nonnegative, and is forced to zero in the final exceptional budget.

Combined with the fixed nonmatching bound, this gives the claimed **global lower bound** 27.

Appendix A. The private lemma $H = 2M$ implies $P \geq 24$. This lemma concerns effective private stars and glues only. It allows arbitrary signed rational coefficients and does not assume that the private data extend to a full PM2 identity.

Index the ten private 3|3 partitions by σ. Let $a_{\sigma,u}$ be the center mass on u's private line, so $a_{\sigma,u} = -2s_{\sigma,u}$. Every vertex has total center mass one. Define

$$B_{uv} = \sum_{\sigma: u,v \in \text{the same half}} a_{\sigma,u},$$
$$H_{uv} = (B_{uv} + B_{vu})/2$$

Each B row sums to two. Each column also sums to two: in every private 3 by 3 glue matrix, the two half-sums of its center masses agree, so the incoming sum at v equals total center mass /2 minus center mass at v, namely $6/2 - 1 = 2$. These identities are valid for signed marginals; the incoming identities are additionally checked in all 90 glue coordinates.

Call u simple if exactly one $a_{\sigma,u}$ is nonzero. That marginal is one, so its B row has two entries one and all others zero. If the two endpoints of a matching edge are both simple, then H on that edge is $0, 1/2$ or 1, and cannot equal two. Thus $H = 2M$ requires a nonsimple endpoint on each of the three disjoint matching edges. At most three vertices are simple, and at least nine private lines have nonzero sum.

For a line with glue vector l and $s = \sum(l)$, its effective stars are $l - s \cdot 1$ and its cost is $f = ||l||_0 + 2||l - s \cdot 1||_0$. Each glue is counted twice and each star twice, giving $2P = \sum f$. If s is nonzero, the possible costs are $5, 7, 8, 9$: with one nonzero glue entry the cost is five; with two, it is eight; with three, the possible star counts are one, two or three. If s is zero, the costs are $0, 6, 9$. Nine nonzero-sum lines therefore cost at least 45. The total is even, and an excess of one over 45 is impossible from these spectra; the next possible even total is at least 48. Ten or more nonzero-sum lines cost at least 50. In either case $P \geq 24$. This proves the lemma used in item 4.

Appendix B. Literal restricted equations for the last case. Write x_j for the scaled coefficient of fixed canonical column j. Each displayed equation is obtained from its literal cell row by keeping the 398 permitted free columns and D_{1437}, and setting every other retained coefficient to zero. The cell equations are homogeneous; the correction columns do not occur in them. The multiplier column gives the exact dual combination from item 8. There are six T rows, fourteen V rows and one g row. Twelve V rows are centered at 2 or 3; the other two are 413 and 443. The twenty-one rows consist of six T rows, fourteen V rows, and one g row.

Cell ID	Kind	Multiplier	Restricted equation
231	T	$-1/2$	$x_{657} + x_{965} = 0$
232	T	$-1/2$	$x_{171} + x_{193} = 0$
233	T	$1/2$	$x_{1281} + x_{1286} = 0$
261	T	$-1/2$	$x_{678} + x_{986} = 0$
262	T	$-1/2$	$x_{411} + x_{433} = 0$
263	T	$1/2$	$x_{1290} + x_{1296} = 0$
276	V	$1/2$	$x_{395} + x_{986} + x_{1369} = 0$
277	V	$1/2$	$x_{396} + x_{678} + x_{1370} = 0$
279	V	$1/2$	$x_{137} + x_{965} + x_{1362} = 0$
280	V	$1/2$	$x_{138} + x_{657} + x_{1363} = 0$
306	V	$1/2$	$x_{395} + x_{433} + x_{1408} = 0$
307	V	$1/2$	$x_{396} + x_{411} + x_{1409} = 0$
309	V	$1/2$	$x_{137} + x_{193} + x_{1398} = 0$
310	V	$1/2$	$x_{138} + x_{171} + x_{1399} = 0$
390	V	$-1/2$	$x_{1296} + x_{1369} + x_{1408} = 0$
391	V	$-1/2$	$x_{1290} + x_{1370} + x_{1409} = 0$
413	V	-1	$x_{137} + x_{138} + x_{139} = 0$
420	V	$-1/2$	$x_{1281} + x_{1362} + x_{1398} = 0$
421	V	$-1/2$	$x_{1286} + x_{1363} + x_{1399} = 0$

Cell ID	Kind	Multiplier	Restricted equation
443	V	-1	$x_{395} + x_{396} + x_{397} = 0$
509	g	1	$x_{139} + x_{397} - x_{1437} = 0$

The sum of these equations with the indicated multipliers is $-x_{1437} = 0$, contradicting its required value one. The complete named keys, full untruncated rows and allowed-column list are stored in the source result; the integrated replay checks the restriction afresh.

Lemma SL8 (The full single-D row identity)**.** *Let* $M = 01 \mid 23 \mid 45$ *and* $U = 01$ *in zero-based coordinates. Admit all* 270 *private columns, all* 120 *nonalternating columns at the two endpoints of* U*, the* 12 *group-*M *alternating columns, and* D_{1437}*. No other retained columns are allowed; in particular* $C = X = 0$*. The twenty-one rows and multipliers in Section S13.1, Appendix B give, on this* 403*-column set,*

$$-x_{1437} + \tfrac{1}{2}(x_{1289} + x_{1291} + x_{1295} + x_{1298}) = 0.$$

Proof. Relative to the displayed 399-column restriction, the added columns are 1289, 1291, 1295, and 1298, precisely the four alternatings in the source array $(M, 01)$. Among the twenty-one rows their only nonzero entries are

Row	Multiplier	1289	1291	1295	1298
233	1/2	1	0	1	0
263	1/2	0	1	0	1

Every other entry of these four columns on the selected rows is zero. All old cancellations are unchanged, proving the identity by the displayed multiplication. The complete literal rows are independently multiplied in `code/check_finite_controls.py`. For a single D in this matching block, the array law is $t_U^M = -2 + 2D$, so the row identity $-D + t_U^M/2 = 0$ gives $-1 = 0$. Coordinate relabeling covers every matching and matching source pair. □

S13.2. **The lower bound 28.**
Scope. The preceding bound is $W \geq 27$.
Obstruction. Exclude matching $R = 24$ and five corrections with $R \leq 23$ by flow and helper costs; $R \geq 22$ handles $e \geq 6$.
Conclusion. $W \geq 28$.

1. Notation and stated exact facts. Retained coefficients are scaled by three. Correction coefficients β_e remain unscaled; write $w_e = 3\beta_e$, so every coordinate vertex has weighted correction degree two. Let K be the set of N-attachment vertices, $m = |K|$, and use (S4.8), with $L_3 = L$ and E the double-loop family; A counts alternating directions and X mixed directions. In a loop block B with source $U = \{u, v\}$, put

$$\zeta_B(u) = Z_B(u) + D_B,$$
$$z_B = \zeta_B(u) + \zeta_B(v),$$
$$\Psi(u) = \sum_B \zeta_B(u),$$
$$\Gamma(u) = \sum_{\text{private lines at} u} s_l$$

The support-safe normalization from the lower 27 proof removes D when at least one endpoint ζ vanishes. Afterwards D sources lie within K, and Z occurs only at active endpoints. The identities and charges used below are stated from the exact cell-law model:

$$\Gamma(u) - \Psi(u) = -1/2,$$
$$\Pi \geq 15 + 2m,$$
$$t_U^G - z_{G,U} = \Omega_G,$$
$$I_e = -w_e + \sum_{\text{source } (B)=e} z_B,$$
$$\sum_{G\text{contains } e} \Omega_G = -w_e,$$
$$2A + L_3 \geq 6 - m,$$
$$\text{divergence } (\delta) = H - \nu - w$$

The generalized slot identity is replayed in all 30 orientations with corrections left arbitrary. The group certificates are homogeneous cell identities, so the displayed Ω law holds for every correction graph. In particular some Ω is nonzero. Also

$$\sum_B z_B = \sum_u \Psi(u) = \sum_e I_e + 6. \quad (1)$$

These are signed equalities; no argument below replaces them by positivity.

For a private line l with sum s and N contribution ν_l, its star vector is $l - s \cdot 1 - \nu_l$. Set $F(l) =$ glues $(l) + 2\cdot$ stars $(l) + 2 \cdot N(l), f(u) = \sum_{\text{lines at} u}$ F(l), and $\pi(u) = f(u)/2 + Z(u) + D(u)/2$. Then $\sum \pi = \Pi$ exactly.

- Inactive vertices have $f \geq 5$. Every active vertex has two distinct N-lines, so $f \geq 8$. If Γ is nonzero, $f \geq 9$.

- An $F = 4N$-line has no glue. An $F = 5$ line with one N coefficient n has glue sum -n or $-n/2$. The corresponding forms are one glue / one star and three glues / no stars. These forms are derived from the V equation and replayed.
- At an active vertex with $f = 9$ and Γ nonzero, there are exactly two N columns, on lines of costs five and four. Four N plus the necessary star cost at least ten. Three N require a nonzero class with three of its distinct lines, costing at least twelve. Zero $-\zeta$ classes contribute cancelling pairs. No additional pure line has cost one or two.
- Equality $\Pi = 15 + 2m$ fixes every vertex charge at its minimum. If some ζ is nonzero, its endpoints have one D of coefficient $1/2$, no Z and $f = 8$. D sources form a matching on K. All Z vanish. In the correction-matching branch this equality forces $A \geq 3$.

The last point is the full equality structure already proved at lower 27. For a fixed source U, distinct D blocks have disjoint h classes. With d nonzero D columns on U, at least $2dN$ columns occur at each endpoint. For $m = 2$ this gives the useful bound

$$\Pi \geq 12 + 4d + (d + z) = 12 + 5d + z, \tag{2}$$

where z counts Z. The terms are inactive charge ten, at least one star at each active center, at least $4dN$ columns, and $d + z$ helpers.

2. Three flow lemmas and a per-group charge. The six coordinate projections, valid for arbitrary corrections, are

$$2\sum_{B} \delta_B(1_W - 1_V)$$
$$= \varrho := \sum_{\text{source } (B)=\{u,v\}} (Z_B(v) - Z_B(u))(e_u - e_v)$$

D cancels. If $Z = 0$, a nonzero flow needs at least three blocks, since the 45 projected columns are nonzero and pairwise nonparallel. Every Z-free nonzero-flow block needs two loop directions, so $C \geq 2b$ in this case. The loop-difference rows also show that no L_3 implies $\delta = 0$.

Single Z costs five loops. Suppose only Z in block B_0 is nonzero. The right side ϱ is supported on its two source vertices, while the projected column c_{B_0} vanishes there and has support on the other four. After removing B_0's flow, the remainder has support two or six. A single other projected column has support four, so at least two other flow blocks are needed. They are Z-free and cost two loops each; B_0 costs at least one L_3 by its S rows. Thus $C \geq 5$, with disjoint charges. D does not affect this.

Shared-source Z blocks. If a vertex u belongs to every Z-block source and $\varrho(u)$ is nonzero, another, Z-free flow block must be nonzero because all those source columns vanish at u. Hence

$$C \geq \text{number of} Z - \text{blocks } + 2. (3)$$

Source-pair balance. Suppose all nonzero endpoint zetas have source $U = \{u, v\}$, and $\delta = 0$. Define

$$\mathrm{NT}_U = \sum_{\text{splits separating } u \text{and } v} (s(u) + s(v))$$

Then

$$\mathrm{NT}_U = w_U - 1. (4)$$

The exact V/h elimination gives $H_U - \nu_U - \mathrm{NT}_U - (\text{ center }_u + \text{ center }_v)/2$ as a linear combination of off-source zetas. With centers one and off-source zetas zero this becomes $H_U - \nu_U = 1 + \mathrm{NT}_U$; the flow equation proves (4). This argument permits arbitrary zero $-\zeta N$ classes at vertices outside U. They cancel through their h-side sums, not by assuming their individual coefficients zero. All 15 source-pair identities are replayed with their off-source residuals.

If an endpoint carries a nonzero source-U class and a nonzero-sum private line in a separating split, its two class N-lines lie in together splits and each costs at least four. The separating line costs at least five. Thus its f is at least 13. The nonzero-class hypothesis is required here.

Per-group charge. For Ω_G nonzero, every edge U of G not contained in K costs one alternating direction in its array or one L_3 in its block. If t_U^G is nonzero, use the array. Otherwise $z_B = -\Omega_G$ is nonzero. There must be exactly one active endpoint; the inactive endpoint has $\zeta = 0$, normalized $D = 0$, and the S-row difference forces an L_3. The arrays and loop blocks are distinct across all (G,U), so the charges add:

$$A + L_3 \geq \sum_{\Omega_G \neq 0} \#\{U \in G : U \text{not contained } \in K\}. (5)$$

3. Five corrections require at least 24 *retained directions.* The signed degree census permits a double-star tree or C_4 plus an isolated pair. Both have an edge of scaled weight exactly two. Let q be the number of nonzero Ω groups. Since one matching can supply only three nonzero edge coordinates, $q \geq 2$. We do not assume that an arbitrary C_4-plus-pair Ω solution uses exactly two groups; kernel additions are allowed.

Every perfect matching has at most floor $(m/2)$ edges wholly inside K. Equation (5) gives the following cover.

- $m \geq 5 : \Pi \geq 25$. At $m = 4, \Pi \geq 23$ and $A + L_3 \geq 2$. At $m = 3, \Pi \geq 21$ and $A + L_3 \geq 4$. These give $R \geq 25$.

- $m = 2$: if $\Pi \geq 20, (5)$ gives $A + L_3 \geq 4$ and $R \geq 24$. At $\Pi = 19$, the equality structure gives $Z = 0$ and at most one D, of coefficient $1/2$. Without D all $3q$ supported arrays are nonzero. With D at most one array can cancel, so $A \geq 3q - 1 \geq 5$. Again $R \geq 24$.
- $m = 1$: if $\Pi \geq 18, (5)$ gives $A + L_3 \geq 6$. At $\Pi = 17$ equality permits neither D nor Z, so every ζ is zero. Effective private stars have support $P_0 \leq \Pi$ and center mass one. Since $P_0 \leq 17$, at most six private lines have nonzero sums (each costs at least five and $2P_0 \leq 34$). Every vertex is therefore simple, so H^0 entries are in $\{0, 1/2, 1\}$. They cannot equal the weight-two correction edge. Flow is nonzero; with $Z = 0$ it costs $C \geq 6$, while $A \geq 3q \geq 6$. This is more than needed.
- $m = 0$: normalize $D = Z = 0$. If private support $P \geq 18, A \geq 3q \geq 6$ gives $R \geq 24$. If $P \leq 17$, the same simple-vertex argument gives nonzero flow and $C \geq 6$, so $R \geq 15 + 6 + 6 = 27$.

Thus $e = 5$ implies $R \geq 24$ and $W \geq 29$. The proof uses only the simple-vertex bound at $P \leq 17$, avoiding an extra dependency on the private support gap.

4. Matching: eliminate the general and one-vertex cases. It remains to exclude $R = 24$ for matching corrections M. Lower 27 already excludes smaller retained support, so this is sufficient.

$N = 0$. Normalization gives $D = Z = 0$. At zero flow the private $H^0 = 2M$ bound gives $P \geq 24$ and $A \geq 3$. At nonzero flow $C \geq 6$, $A \geq 3$ and $P \geq 15$, so $R = 24$ would force $(P, A, C, X) = (15, 3, 6, 0)$. The private-15 star-block argument (S5.2) instead gives $A \geq 6$. Thus no $N = 0$ identity has retained support 24.

All zetas zero with $N > 0$. Effective private support is at most Π, center masses stay one, H equals its effective value, and ν vanishes. Also $A \geq 3$, so $R = 24$ gives $\Pi \leq 21$. The private $H = 2M$ lemma would require effective support at least 24. Hence nonzero flow forces $C \geq 6$, and $\Pi \geq 17$ gives $R \geq 26$. This case is excluded.

$m \geq 4$. At $m \geq 5, \Pi \geq 25$. At $m = 4, \Pi \geq 23$; equality gives $A \geq 3$, while $\Pi \geq 24$ still needs $A + L_3 \geq 1$. Thus $R \geq 25$.

$m = 1$. $D = 0$, all Z attach at one vertex u, and some Z is present. Let z be their count. The two matching edges avoiding u stay nonzero, so $A \geq 2$. For $z = 1, \Pi \geq 18$ and $C \geq 5$ give $R \geq 25$. For $z \geq 2$, distinct classes and blocks give $\Pi \geq 14 + 3z$ and $C \geq z$. Values $z \geq 3$ are excluded. At $z = 2, R = 24$ forces $\Pi = 20, A = 2, C = 2, X = 0$. Both Z sources must be the matching edge containing u, with their signed sum exactly two: otherwise another isolated sum remains nonzero. Thus $\varrho(u) = -2$ and (3) gives $C \geq 4$, a contradiction.

We may now assume $m = 2$ or 3 and some ζ nonzero.

5. Matching: two attachment vertices. Let $K = U = \{u, v\}, d$ count D and z count $Z, h = d + z$. All D sources are U. The four inactive vertices and

the two active line charges give $\Pi \geq 18 + h$. At least one matching edge avoids K, so $A \geq 1$; if $z = 0$, all helpers have source U and $A \geq 2$.

5.1 No Z. Here $d \geq 1$. Nonzero flow would cost $C \geq 6$ and give $R \geq 19 + 2 + 6 > 24$, so $\delta = 0$. For $d \geq 3$, (2) gives $\Pi \geq 27$. For $d = 2$, (2) gives $\Pi \geq 22$, forcing $\Pi = 22, A = 2, C = X = 0$. Equality leaves no glue at either active endpoint, hence $\Gamma = 0, \Psi = 1/2$ and the signed sum of D is $1/2$. Then $I_U = -2 \cdot 1_M(U) + 1$ is nonzero, and at least three isolated sums are nonzero, contradicting $A = 2$.

For $d = 1$ write its coefficient a. Balance (4) requires $\mathrm{NT}_U = +1$ or -1, so one endpoint has $f \geq 13$; the other has $f \geq 8$. Thus $\Pi \geq \mathrm{ceil}\,(10+13/2+8/2+1) = 22$. The budget forces $A = 2, C = X = 0$ and $\Pi = 22$. The slot law now pins U in M and $a = 1$. Both endpoint Γ values are $1/2$, so both have $f \geq 9$; the only possible split is $(13, 9)$. This order of deductions excludes the otherwise possible $(14, 8)$ split.

At the $f = 9$ endpoint all lines belong to its two source-U class lines, so its separating sum is zero. At the other endpoint the two together class lines cost four each and a separating line costs five, exhausting $f = 13$. The cost-four lines have no glue. Therefore its entire $\Gamma = 1/2$ lies on the separating line, so $\mathrm{NT}_U = 1/2$, contradicting (4)'s value one. This does not assume extra cancelling N columns are absent; any that fit the line budget cannot create another glue-bearing line.

5.2 At least one Z. Now $C \geq 1$. Coverage $2A + L_3 \geq 4$ implies $A + C \geq 3$, hence $\Pi \leq 21$. Consequently $h \leq 3$, and (2) excludes $d \geq 2$. A single Z is impossible because $\Pi \geq 19, A \geq 1, C \geq 5$. Only $(d, z) = (0, 2), (0, 3), (1, 2)$ remain.

For $h = 3, \Pi = 21$ exhausts the inactive, active-line and helper charges. Both endpoints are glue-free: $\Psi(u) = \Psi(v) = 1/2$ and $\sum_B z_B = 1$. If $C \geq 2$, the budget forces $A = 1$. The sole alternating column is on an unaffected matching edge and has coefficient -2, so (1) instead gives $\sum_B z_B = 4$. If $C = 1$, the two Z must share one block, so the case is $(1, 2)$ with every helper on U. The source sum one leaves I_U nonzero in addition to the unaffected matching edges, forcing $A \geq 3$. Both cases contradict $R = 24$. This is a signed argument throughout.

For $(d, z) = (0, 2), \Pi$ is 20 or 21.

- At $\Pi = 20$ both active line charges are four, so $\Gamma = 0$ and there is one Z of coefficient $1/2$ at each endpoint. All three matching isolated sums remain nonzero, so $A \geq 3$. The budget forces $C = 1$ and both Z in one source-U block. The sole loop must be the double loop; there is no L_3, hence $\delta = 0$. Glue-freeness gives $\mathrm{NT}_U = 0$, contradicting (4).
- At $\Pi = 21$ the only budgets are $A = 1, C = 2$ or $A = 2, C = 1$, with $X = 0$. In the first, U is not a matching edge, and the two Z are each forced to coefficient two on the two matching sources incident to K. Thus $\Gamma(u) = \Gamma(v) = 3/2$. The charge budget forces $f = 9$ at both vertices.

Their class N coefficients are -2, whose cost-five line sums are two or one, never $3/2$. In the second budget, both Z share a source-U block. Its sole double loop forces equal Z coefficients; the slots force each to one and U in M. With no L_3 the flow vanishes. Balance requires a separating line, so $f \geq 13$ at one endpoint and $f \geq 9$ at the other. This gives $\Pi \geq 10 + (13 + 9)/2 + 2 = 23$, a contradiction.

This exhausts all $m = 2$ helpers, including their boundary budgets.

6. Matching: three attachment vertices. $\Pi \geq 21$. If $\Pi \geq 23$, coverage $2A + L_3 \geq 3$ requires at least two other directions, giving $R \geq 25$. At $\Pi = 21$ equality forces $A \geq 3$, so $A = 3$ and $C = X = 0$. Some ζ is nonzero, and the equality structure leaves exactly one D of coefficient $1/2$ with glue-free endpoints U; the third active vertex has only zero $-\zeta$ classes. No loops implies $\delta = 0$. Equation (4) still applies with that third vertex present, and contradicts $\mathrm{NT}_U = 0$.

At $\Pi = 22$ only $(A, C, X) = (2, 0, 0)$ or $(1, 1, 0)$ can meet coverage.

In the first case $Z = 0$ and every D source lies inside K. There must be a matching edge $U = \{u, v\}$ inside K; otherwise three unaffected matching slots force $A \geq 3$. Write w for the third active vertex. The two other matching edges have isolated $\sum -2$ and exhaust $A = 2$. Hence the signed sums of D are one on U and zero on every other source. Thus

$$\Gamma(u) = \Gamma(v) = 1/2, \Gamma(w) = -1/2$$

All three active vertices have $f \geq 9$. If d is total D support, disjoint charges give $\Pi \geq 3 \cdot (5/2) + 3 \cdot (9/2) + d = 21 + d$. Since $\Pi = 22$, there is exactly one D, on U, of coefficient one, and $f = 9$ at each active vertex. The two endpoint class lines are together lines and exhaust their line budgets, so $\mathrm{NT}_U = 0$. Yet $C = 0$ gives $\delta = 0$ and (4) requires $\mathrm{NT}_U = 1$. This argument permits signed off-source cancellations and overlapping private lines; neither can evade the count d.

In the second case coverage forces the sole loop to be L_3, not a double loop. It enters exactly one S row. That row must contain exactly one Z, and every other Z would violate its own S row. Thus there is exactly one Z globally, which requires $C \geq 5$. This contradicts $C = 1$.

Every matching $R = 24$ case is excluded, so the matching case has $R \geq 25$. The five-edge case above has $R \geq 24$. For $e \geq 6$, the general retained bound $R \geq 22$ from Section S6.1 suffices. The correction census excludes $e < 3$ and $e = 4$, and every three-edge correction graph is a matching. Consequently all correction supports satisfy

$$W = e + R \geq \min\{3 + 25,\ 5 + 24,\ 6 + 22\} = 28.$$

This proves $W^*(6, 2) \geq 28$.

S13.3. **The lower bound 29.**
Scope. The preceding bound is $W \geq 28$.
Obstruction. Raise matching retained support to 26 and the six-correction bound to 23; test the tight private core against actual star-bearing blocks.
Conclusion. $W \geq 29$.

1. Established inputs and notation. Use the notation, normalization, exact identities and support counts of the lower-bound -28 component above. Thus

$$R = \Pi + A + C + X, \Pi = \text{stars } + \text{glues } + N + Z + D, C = L_3 + E,$$
$$\Pi \geq 15 + 2m, \Gamma(u) - \Psi(u) = -1/2,$$
$$t_U^G = \Omega_G + z_{G,U}, \sum_{G\text{contains } e} \Omega_G = -w_e,$$
$$\sum_B z_B = \sum_e I_e + 6,$$
$$A + L_3 \geq \sum_{\Omega_G \neq 0} \#\{U \in G : U\text{not contained } \in K\}$$

K is the set of N-attachment vertices; $m = |K|$. In the matching case, $w = 2 \cdot 1_M$. A is alternating support, C loop support, X mixed support. For a private line, $F =$ glues $+2\cdot$ stars $+2 \cdot N$; f(u) is its sum at u. An inactive vertex has $f \geq 5$. An active vertex has $f \geq 8$, or $f \geq 9$ if Γ is nonzero. Equality $\Pi = 15 + 2m$ permits no Z and only disjoint D columns of coefficient $1/2$, with glue-free endpoints; in the matching case it forces $A \geq 3$. All $-\zeta$-zero matching identities with $N > 0$ have $R \geq 26$: otherwise effective private support is below 24, hence nonzero flow costs $C \geq 6$ and $\Pi \geq 17, A \geq 3$.

The normalization is support-nonincreasing and preserves the correction graph and N. Since matching $R \geq 25$ is already proved, any hypothetical $R = 25$ identity, before or after normalization, admits no support-decreasing exact kernel move. This minimality is used only for the explicit four-column relation

$$(D, Z_u, Z_v, E) \to (D + a, Z_u - a, Z_v - a, E + a). \quad (1)$$

Whenever $Z = 0$, nonzero flow costs $C \geq 6$. A single nonzero loop column with $Z = 0$ is impossible by its S rows. A single Z globally costs $C \geq 5$. Every block containing Z needs a loop column. Source-pair balance says that if every nonzero ζ has source $U = \{u, v\}$ and flow vanishes, then

$$\mathrm{NT}_U := \sum_{\text{private splits separating } u,v} (s(u) + s(v)) = w_U - 1. \quad (2)$$

If an endpoint has a nonzero U-class and a nonzero-sum separating line, its class lines cost $4+4$ and that extra line costs 5, so $f \geq 13$. Zero $-\zeta$ classes at other vertices are permitted in (2).

All these inputs have literal rational row certificates in the lower 28 checkpoint. The proof also uses the private P15 incidence checks and the extended fixed 21-row dual already pinned there. Neither the two-private- partition theorem nor the six-star / all-correction theorem is needed.

2. A signed kernel lemma forces one matching group. **Lemma.** Any nonzero linear dependence among distinct perfect-matching incidence vectors on six vertices has at least six nonzero coefficients.

Distinct perfect matchings share at most one edge. In a dependence, every edge of a supported matching must occur in another supported matching. Thus every matching needs at least three other matchings, one for each edge, and support below four is impossible.

With four matchings, an edge shared by three would leave a containing matching's other two edges to be covered by the sole remaining matching, which cannot share two edges with it. Thus no triple-shared edge exists. Every pair must intersect, each intersection edge occurs exactly twice, and its coefficients are negatives of one another. A triangle in this intersection graph makes every coefficient zero.

With five matchings, first suppose no edge occurs three times. Then every matching has exactly three neighbors in the intersection graph. This would be $a3$-regular graph on five vertices, impossible by parity. An edge cannot occur more than three times in K6. If three matchings G_1, G_2,G3 share an edge, their other two edges must each be covered by the two remaining matchings H1,H2, one edge from each. The six other edges of G_1, G_2,G3 are distinct. Each Hi therefore contains one from each Gj. Their edge equations force all three Gj coefficients to be equal and opposite to either Hi coefficient. The shared-edge equation then reads $3a = 0$, a contradiction. This proves the lemma over Q (indeed over R).

For matching corrections, $\Omega + 2 \cdot e_M$ is in this kernel. Unless it is zero, the support q of Ω is at least five. Every matching has at most floor $(m/2)$ edges inside K. The per-group charge would therefore give

$$R \geq 15 + 2m + 5 \cdot (3 - \text{floor } (m/2))$$

For $m = 0,...,6$ these bounds are respectively $30, 32, 29, 31, 28, 30, 27$. Consequently

$$\text{matching } R \leq 26 \Longrightarrow \Omega_M = -2, \text{all other } \Omega_G = 0. \quad (3)$$

For the rest of the matching proof assume $R = 25$ and use (3). Each array in M has total $z_B - 2$; each array outside M has total z_B. Every nonzero array total requires its own alternating column. These arrays are disjoint, including arrays with the same source edge in different groups.

3. Additional small local facts. **Two helper counts.** If $Z = 0$ and vertex u meets d_u different D blocks, its h classes require at least $2 \cdot d_u N$ columns, and its centered star mass requires at least one star. Hence

$$\pi(u) \geq 1 + (5/2) \cdot d_u. \tag{4}$$

For $m = 2$ and d D columns, this gives $\Pi \geq 12{+}5d{+}z$ even when Z is present: normalization makes both zetas of every D block nonzero. Z is an additional support charge; it is not assumed to require additional N when it shares a D block. Also for $m = 2, \Pi \geq 18 + d + z$. For $m = 3, \Pi \geq \text{ceil } (19.5 + d + z)$.

Equal Z coefficients. If $Z_u = Z_v = a \neq 0$ in a block with $D \neq 0$, move (1) deletes both Z and leaves at most D and E, saving at least one direction. If that block has $D = 0$ and its only loop column is E, the S rows give $E = -a$, and (1) replaces its three Z,Z,E terms by one D. Thus an $R = 25$ matching identity with Z present cannot have $C = 1$. All its Z would lie in one block. A sole L_3 has only one nonzero S incidence, so the other S row forces one endpoint Z to vanish; exactly one Z then remains and the single-Z bound requires $C \geq 5$. A sole E instead forces equal endpoint Z and the support-decreasing move just described.

Z in one block. Suppose all Z are in one block B with source $\{u, v\}$, and $C \leq 4$. The coordinate projection of the flow has right side $(Z_v{-}Z_u)\cdot(e_u{-}e_v)$. The column c_B is supported on the other four coordinates. If $Z_u \neq Z_v$, after subtracting B's flow the residual has support two or six. A single other flow column has support four. At least two Z-free flow blocks are necessary, costing four loops in addition to B's one, a contradiction. Hence $Z_u = Z_v$. The projection is now zero; nonzero flow needs at least three distinct blocks and again costs at least $1 + 2 + 2 = 5$. Therefore all flow vanishes. This argument also allows B's flow to be zero and arbitrary signs.

The one-N line of cost six. Let its unique N coefficient be $n \neq 0$, in position i. For $F = 6$, the only possible support counts are two glues and one star: zero glues would produce exactly one star and $F = 4$. Solving $S = l - s \cdot 1 - n \cdot e_i$ with those supports forces, up to the other position j, $l = n \cdot e_i - n \cdot e_j$, $s = 0$, and $S = -n \cdot e_j$. Thus the line sum is zero. The existing $F = 5$ forms have sum -n or $-n/2$; $F = 4$ has no glue.

In particular, suppose an active vertex has exactly one nonzero ζ class, of value $a = 2$, and no other helper there. Then $\Gamma = 3/2$ and $f \geq 11$. If $f \leq 10$, there are exactly two N columns: three in a nonzero class use three lines and cost at least 12; four N plus a star and a glue cost at least 11. Each of the two N coefficients is -2. The only two-line costs up to 10 are $(4, 4), (4, 5), (4, 6), (5, 5)$, with total glue sums in $\{0, 1, 2, 3, 4\}$, never $3/2$. A third nonempty line would cost at least 5 or 6.

For later equality cases, $f = 8$ and no glue forces two cost $4N$-lines. Each has either one star and one N, or no star and two N in the same V position with opposite coefficients. Two star-free lines contradict center mass one. One line of each kind would give three N. The two cancelling

columns in the star-free line belong to different prefix classes, as the literal V incidences verify. Both classes would need partners elsewhere; the third N cannot supply both. Thus at least four N would be needed, contradicting the cost. In the remaining case each line has one N and one star. They cannot be a zero $-\zeta$ cancelling pair because their star sum would vanish; they belong to one nonzero class and center mass one makes that class value $1/2$. At $f = 9$ with $\Gamma \neq 0$ there are exactly two N columns on cost 5 and cost 4 lines; their only glue-bearing line belongs to that class.

4. Matching: $N = 0$, all $-\zeta$-zero, and $m \geq 4$. For $N = 0$, zero flow requires private support $P \geq 24$ and $A \geq 3$. Nonzero flow costs $C \geq 6$. $R = 25$ would then give $P \leq 16$. The signed private line spectrum is $\{5, 7, 8, 9\}$ for a nonzero marginal and $\{0, 6, 9\}$ for a zero marginal. $P \leq 16$ leaves exactly six nonzero marginal lines, each $\sum -1/2$, and no nonempty zero-marginal line. $P = 16$ would require one cost 7 line. Its two entries outside $\{0, \pm 1/2\}$ force two other crossing lines to have cost above 5, impossible. Thus $P = 15$. The stronger gap $P = 15$ or $P \geq 18$, used in later components, is Lemma SL3.

At $P = 15$, every star is positive and every line has cost 5. If there is a three-glue line, its positive entry forces a full crossing line, so all six vertices use that same split. Its stars occupy distinct loop blocks, and there are at least six. Otherwise all lines are single-glue lines: three glue generators give twelve positive stars and no block receives more than two. In either case $A \geq 6$ by the $g + g - S - S$ block identities. This incidence statement is already checked on all ten splits and the 216 prescribed generator triples. Hence $R \geq 15 + 6 + 6 = 27$, contradiction. This proves the needed $N = 0$ exclusion without the stronger partition theorems.

All $-\zeta$-zero $N > 0$ was excluded in item 1. Suppose some ζ is nonzero. For $m \geq 5, \Pi \geq 25$ and coverage requires at least one additional direction when $m = 5$; $m = 6$ gives $\Pi \geq 27$. At $m = 4, \Pi \geq 23$. Equality forces $A \geq 3$. $\Pi \geq 25$ is impossible by $2A + L_3 \geq 2$. At $\Pi = 24$ the sole remaining direction must be $A = 1, C = X = 0 : A = 0$ would require at least two L_3 directions. The single alternating column leaves two zero M arrays. Thus $Z = 0$. Two M arrays must be zero, requiring $D = 1$ in each of their two blocks, whose four endpoints are exactly K; no other D block can have a nonzero array. Each active Γ is $1/2$ and has $f \geq 9$. This gives $\Pi \geq 2 \cdot 2.5 + 4 \cdot 4.5 + 2 = 25$, contradiction.

5. Matching: one attachment vertex. $D = 0$. Let z count Z columns. If $z \geq 3$, their disjoint h classes give $\Pi \geq 14 + 3z, C \geq z, A \geq 2$, already $R \geq 28$. Only $z = 1$ or 2 remains.

For $z = 1, \Pi \geq 18, C \geq 5, A \geq 2$ force $\Pi = 18, C = 5, A = 2, X = 0$. The two unaffected matching slots exhaust A, so the Z source is the matching edge at u and its coefficient is 2. All five inactive vertices have $f = 5$ and u has $f = 9$. But its $\Gamma = 3/2$ contradicts item 3'$sf \geq 11$.

For $z = 2, \Pi \geq 20$ and $C \geq 2$. At most one Z block can be a group-M block; the other has a nonzero array outside M. The two unaffected M arrays therefore give $A \geq 3$. Equality $R = 25$ forces $\Pi = 20, A = 3, C = 2, X = 0$. The unique group-M Z is 2, cancelling its array; write b for the other Z. At u there are at least four N columns and one star, so $f(u) \geq 10$. The five inactive vertices each cost at least 5, and their first possible excess is 2. Since $\sum f = 2 \cdot (\Pi - z) = 36$, they all have $f = 5$ and $f(u) = 11$. Thus u has exactly four N, one star and one glue. That glue crosses an inactive cost 5 line, so its coefficient is $\pm 1/2$. Therefore $2 + b = \Gamma(u) + 1/2$ is 0 or 1, giving $b = -2$ or -1. Every N coefficient is now an integer: each class has just one column on each side. The single glue creates two effective-star entries of value $\pm 1/2$. Subtracting integer N contributions leaves both nonzero, contradicting the single actual star. This excludes $m = 1$.

6. Matching: three attachment vertices. First let $Z = 0$ and d count D. The case $d = 0$ is the previously excluded all $-\zeta$-zero case. Nonzero flow cannot fit $R = 25$, since $\Pi \geq 21, A \geq 2, C \geq 6$; hence flow is zero when needed below.

For $d = 1$, balance (2) forces a separating line at an endpoint of its source U. Its f is at least 13, the other endpoint at least 8, the third active vertex at least 9, and the inactive sum at least 15. Including $D, \Pi \geq$ ceil $((13 + 8 + 9 + 15)/2 + 1) = 24$. At least two M arrays remain, so $R \geq 26$. For $d = 2$ with a common source, the endpoints each have charge at least 6 by (4); the third active vertex costs 4.5 and the inactive vertices 7.5. Thus $\Pi \geq 24$ and $A \geq 2$. If the two sources differ, their shared endpoint costs 6 and the two other endpoints at least 4.5 each, giving $\Pi \geq 23$. At most one D can cancel an M array, while an off-M D supplies an extra nonzero array; thus $A \geq 3$. With $d \geq 3, (4)$ summed over K gives $\Pi \geq$ ceil $(7.5 + 3 + 5d) \geq 26$. All $Z = 0$ cases are excluded.

Now let $Z > 0$. $\Pi \geq 22$, because equality at 21 has no Z. A single Z costs $C \geq 5$ and is impossible. Write $h = d + z$; the bound $\Pi \geq$ ceil $(19.5 + h)$ shows that at $\Pi = 22$ there are exactly two Z, no D. The total line budget has just one unit above three inactive costs 5 and three active costs 8. At least two active Γ values must be zero. The two Z must therefore attach at different vertices and both equal $1/2$; the third vertex has $\Gamma = -1/2$. No M array can be cancelled by this total helper mass 1, so $A \geq 3$. With $C \geq 1$ this gives $R \geq 26$.

If $\Pi \geq 23$, at most two directions remain. $C = 1$ is ruled out by the kernel move in item 3. The only possible budget is $A = 0, C = 2$. If K contains an M edge, another M edge avoids K and forces an alternating direction. Otherwise K contains one endpoint of each M edge. To cancel all three M arrays then requires a Z in each of their three separate blocks, hence $C \geq 3$. This excludes $m = 3$, including all mixed-column budgets.

7. Matching: two attachment vertices, no Z. Write $K = U, d = \#D$. Each D has source U. $\Pi \geq 12 + 5d$ excludes $d \geq 3$. If $d = 2, \Pi \geq 22$. At least

one of the D blocks is outside M, or U itself is not an M edge. Formula (3) therefore gives $A \geq 3$. $R = 25$ forces $\Pi = 22, A = 3, C = X = 0$. Equality in the count $12 + 5d$ leaves no glue at either active endpoint. Balance (2) instead requires $\mathrm{NT}_U = +1$ or -1. Contradiction.

Let $d = 1$. Nonzero flow costs $C \geq 6$ and cannot fit; balance (2) forces $f \geq 13$ at one endpoint, so $\Pi \geq 22$. If the D block is outside M, all three M arrays and its own extra array are nonzero, giving $A \geq 4$ and $R \geq 26$. This also excludes U outside M, since no M block then has source U. Thus U belongs to M and the D is in the block (M,U), with coefficient a. At most three alternating columns remain. No outside-M array can be active: it would have zero total and need two columns, in addition to the two unchanged M arrays. There can be no loop column: $C \leq 1$ and $Z = 0$, whereas every L_3 has a nonzero S incidence and E has two. A sole nonzero loop cannot satisfy those homogeneous S rows. Therefore $C = 0$.

If $X = 0$, the previously checked 21-row dual on all private columns, N attached anywhere in U, all group-M alternatings and this D reads

$$-a + t_U^M/2 = 0$$

But $t_U^M = -2 + 2a$, making its left side -1. If $X = 1$, the budget forces $\Pi = 22, A = 2, a = 1$. Both Γ values are $1/2$ and both $f \geq 9$. Balance forces a separating line, so the only charges are $(13, 9)$, with inactive $f = 5$. The cost 13 endpoint has two glue-free cost 4 together lines and a cost 5 separating line. Its separating sum equals its entire $\Gamma = 1/2$. The cost 9 endpoint has only together lines. Thus $\mathrm{NT}_U = 1/2$, contrary to (2)'s value 1. This also excludes the one-mixed-column boundary.

8. Matching: two attachment vertices, Z present. At least one M edge avoids K, so $A \geq 1$. Coverage $2A + L_3 \geq 4$ and $C \geq 1$ give $A + C \geq 3$, hence $\Pi \leq 22$. A single Z has $\Pi \geq 20$ and $C \geq 5$, already $R \geq 26$. Thus $z \geq 2$. Since $\Pi \geq 18 + d + z$ and $\Pi \geq 12 + 5d + z$, only $d \leq 1$ and $h = d + z \leq 4$ remain. $C = 1$ is excluded by item 3; hence $C \geq 2$.

$h = 4$**.** $\Pi = 22$ and all inactive $f = 5$, active $f = 8$. The active Γ values vanish, giving $\sum z_B = 1$. The remaining budget would require $A = 1, C = 2$. This implies U is outside M; otherwise two M edges avoid K and require two alternating columns. Its sole alternating column is on the unaffected M edge and has coefficient -2, so $\sum I = -2$ and the exact total law instead gives $\sum z_B = 4$. Contradiction.

$h = 3, \Pi = 21$**.** The same equality forces active $f = 8$, no glue and one nonzero class of value $1/2$ at each endpoint. Three Z without D would occupy too many classes. With one D and two Z, all must share one block, and the Z coefficients are equal to $1/2 - D$. Move (1) saves at least one direction, contradicting the proved matching lower bound.

$h = 3, \Pi = 22$**.** The remaining budget is $A = 1, C = 2$. K must meet two different M edges, and each corresponding M array must be cancelled. This

needs a Z of coefficient 2 in each of those two blocks, since D has source K, outside M. Their other endpoints are inactive, so no additional Z can share either of those blocks. With $(d, z) = (0, 3)$, the third Z therefore lies in another block and gives an extra nonzero array. With $(1, 2)$, the D gives that extra array. Neither can fit $A = 1$, which is already on the unaffected M edge. Thus $h = 3$ is excluded.

It remains that $d = 0, z = 2$, with $20 \leq \Pi \leq 22$. For $\Pi = 20, 21, 22$ the respective upper bounds on A are $3, 2, 1$. The two Z may share a block.

If their blocks differ, each outside-M Z block gives an extra nonzero array. With one such block $A \geq 3$; with two $A \geq 5$. If both are in M, their attachment vertices are on different M edges. $A \leq 2$ forces at least one coefficient to equal 2, and $A = 1$ forces both to equal 2. The value 2 endpoint has $f \geq 11$ by item 3, while the other endpoint has $f \geq 8$. Hence $\Pi \geq$ ceil $(10+2+(11+8)/2) = 22$. This excludes $A \leq 2$ at $\Pi \leq 21$; $A = 1$ at $\Pi = 22$ forces both endpoint costs ≥ 11, hence $\Pi \geq 23$. The same argument excludes the one $-M/$ one-outside-M configuration at $A = 3$ and $\Pi = 20$, since its M coefficient must equal 2 to cancel that array. If both Z attach to the same vertex, their four N plus the other active vertex's cost 9 already force $\Pi \geq 22$, and their distinct blocks give $A \geq 3$. Thus no case was lost by the endpoint bounds above.

If both Z share a block outside M, the three M arrays remain nonzero. If they share a block in $M, A \leq 2$ requires their sum to equal 2. All Z are then in one block and the remaining loop budget is at most 3; item 3 gives equal coefficients and zero flow. Thus both equal 1. Both endpoint costs are at least 9, forcing $\Pi \geq 21$. At $\Pi = 21$ they both have $f = 9$, only together class lines, so $\mathrm{NT}_U = 0$; (2) requires 1.

The only budget still to consider is $\Pi = 20, A = 3, C = 2, X = 0$. Equality fixes inactive $f = 5$, active $f = 8$, with one Z of coefficient $1/2$ at each endpoint. If they share an M block, the one-block lemma makes the flow zero, and glue-freeness contradicts (2). A shared outside-M block would need $A \geq 4$. For distinct blocks, one outside M would need $A \geq 4$ because its group-M companion has coefficient $1/2$, not 2; two outside M need $A \geq 5$. Thus the only remaining placement is in two distinct group-M blocks. The two loops must lie one in each Z block. Their projected columns are constant on both coordinates of every M edge. The right side of the projection identity is instead opposite nonzero values on each of the two affected M edges (one endpoint has $Z = 1/2$, the other zero). This is impossible. All $m = 2$ cases are excluded.

We have excluded every matching identity with $R = 25$. The earlier bound excluded smaller R, so matching retained support is at least 26.

9. Six corrections cannot have retained support 22. Six correction edges require at least two nonzero Ω groups, $q \geq 2$. For $N > 0$ the per-group charge gives, respectively at $m = 1, 2, 3$,

$$R \geq 17 + 6 = 23, R \geq 19 + 4 = 23, R \geq 21 + 4 = 25$$

At $m \geq 4, \Pi \geq 23$ alone suffices. Thus $R = 22$ would force $N = 0$ and normalized $Z = D = 0$. Then $A \geq 3q \geq 6$, so $P \leq 16$. Nonzero flow would cost $C \geq 6$ and give $R \geq 27$; hence flow is zero and $H = w$.

The private line count makes every vertex simple and every nonempty zero-marginal line impossible: its cost 6 exceeds the slack above six cost 5 lines. Therefore H entries are in $\{0, 1/2, 1\}$. Also $A \leq 7$ implies $q = 2$. Two perfect matchings supplying six correction edges must be edge-disjoint; their union is a six-cycle. The correction degree 2 at each vertex and $H \leq 1$ force weight 1 on every cycle edge.

The directed private formula $H_{uv} = (B_{uv} + B_{vu})/2$ now forces each simple vertex's two halfmates to be its cycle neighbors. Opposite vertices are the only active centers in each of three complementary 3|3 splits. Since zero-marginal lines vanish, each such private matrix has a single glue of coefficient $-1/2$ and four positive stars. The six center-halfmate source pairs are distinct, and each center's two stars occupy distinct blocks. Thus all twelve stars occupy twelve distinct charged blocks, each requiring an alternating direction. This contradicts $A \leq 7$.

This last prescribed six-cycle incidence is checked directly against the literal V equations, H rows, and all star-bearing block rows. No six-star subclass assumption enters it: this private core has twelve stars.

Global conclusion. The preceding cases give retained support at least 26 for three corrections, 24 for five, and 23 for six. For at least seven corrections the general bound $R \geq 22$ suffices. The correction census excludes fewer than three and exactly four corrections. Hence

$$W = e + R \geq \min\{3 + 26, 5 + 24, 6 + 23, 7 + 22\} = 29.$$

This proves $W^*(6, 2) \geq 29$ using only the preceding lower bound and the local lemmas.

S13.4. **The lower bound 30.**
Scope. The preceding bound is $W \geq 29$.
Obstruction. The correction count fixes the retained budget. Nonmatching cuts and matching helper cases end in physical source-flow contradictions.
Conclusion. $W \geq 30$; each strict support decrease uses only the preceding bound.

1. stated notation and exact identities. Use the canonical model and normalization of the preceding lower-bound 29 component.

Use (S4.8), with $L_3 = L$, and the previously proved bound $\Pi \geq 15 + 2m$ from S13.1, item 2. K is the set of N-attachment vertices, $m = |K|$. Lowercase d,z count D,Z columns; these are support counts, not sums of their coefficients. For a block B with source $U = \{u, v\}$, put

$\zeta_u = Z_u + D, \zeta_v = Z_v + D, \beta_B = Z_u + Z_v + 2D$. Normalization removes D if either endpoint ζ is zero. Consequently each surviving D uses a nonzero class at both endpoints; Z can attach only in K. At a private line let

$F = g+2s+2n$ count its glues, stars and N; f(u) is the sum of F at u. Each private glue is counted twice, so

$$2\Pi = \sum_u f(u) + 2(d+z)$$

An inactive vertex has $f \geq 5$, with first strict increase at least 2. An active vertex has $f \geq 8$, or $f \geq 9$ if its total glue marginal Γ is nonzero. $\Gamma(u) = \Psi(u) - 1/2$, with $\Psi = \sum_B \zeta_u(B)$. At $f = 8$ there are no glues, precisely two N in one nonzero class, and its value is $1/2$. At $f = 9$ with $\Gamma \neq 0$ there are two N, on a cost 5 and a cost 4 class line; all glues at that vertex lie on the cost 5 line. The earlier proof's different-class collision argument is included in this equality statement.

Every vertex has actual star mass 1, hence at least one actual star even when its coefficients have arbitrary signs. Indeed the V-line formula is $S = l - s \cdot 1 - \nu$, so its star sum is $-2s - \sum(\nu)$. Summing at u and using the two h-side sums gives $-2\Gamma(u)+2\Psi(u) = 1$. This is consistent with the stated star-center dual, whose right side is $24 \cdot (6 \cdot \text{starCenterIndicator } - \text{starIndicator })$ before normalizing the retained coefficients by 3. The right side vanishes on all mixed columns.

Each class is (matching group G, source pair U, attachment u in U). Its four N columns lie on four different private lines, all in halves containing U. Each of its two h sides sums to $-\zeta$. A nonzero class needs at least two N; with precisely two, each coefficient is $-\zeta$.

The general array and group identities are

$$t_U^G = \Omega_G + \beta_{G,U},$$

$$\sum_{G \text{contains } e} \Omega_G = -w_e,$$

$$\sum_B \beta_B = \sum_e I_e + 6,$$

$$A + L_3 \geq \sum_{\Omega_G \neq 0} \#\{U \in G : U \text{not contained } \in K\},$$

$$2A + L_3 \geq 6 - m$$

Every nonzero array total requires its own alternating column. Array supports are disjoint, including different groups with the same source. For matching corrections $w = 2 \cdot 1_M$, the proved signed-incidence lemma forces $\Omega = -2 \cdot e_M$ whenever $R \leq 26$. Thus the three M-array totals are $\beta - 2$; every other array total is β.

The projection of the flow and the effective-private flow identity are

$$2\sum_B \delta_B c_B = \sum_B (Z_v - Z_u)(e_u - e_v),$$

$$\operatorname{div}(\delta)_e = H^0_e + \sum_B \beta_B(1[e=U] + 1[e=V]/2 + 1[e=W]/2) - w_e.(F)$$

Here $c_B = 1_W - 1_V, U, V, W$ are the three disjoint pairs of B, and H^0 is computed from effective stars obtained by eliminating N from V. Its glue-marginal form is $H^0_{uv} = (B_{uv} + B_{vu})/2$, where

$$B_{uv} = -2 \sum_{\text{private halves at} u \text{containing } v} \text{line } \textstyle\sum(u)$$

The 15 literal rational certificates for (F) are independently recomputed by the checker listed in Section S12.8. A simple inactive vertex has one line of marginal $-1/2$, so its B row has two entries 1 and all others 0. A glue-free vertex has a zero B row. H^0 need not be nonnegative in general; the proof only uses these explicitly justified special cases.

At zero flow, summing (F) gives the signed identities

$$\sum_e w_e = 6, \sum_G \Omega_G = -2.(M)$$

To see this without any positivity assumption, each B row sums to $-4\Gamma(u)$, so $\sum_e H^0_e = -2\sum_u \Gamma(u) = 6 - 2\sum_B \beta_B$. Each helper contributes $\beta \cdot (1 + 1/2 + 1/2) = 2\beta$ to the edge sum. The terms cancel. Summing the group incidence law then gives $3\sum_G \Omega_G = -6$. All three uses of (M) in item 3 have zero flow.

The literal $g + g - S - S$ row combination in every block is

$$\sum_{\text{four block stars}} \text{coefficient} + \sum_{\text{four block alternatings}} \text{coefficient} = \beta_B.(B)$$

The 45 block-star supports partition the 180 stars, and their alternating supports partition the 180 alternating columns. Thus when $\beta = 0$ and actual stars are nonnegative, k blocks with positive star sum force $A \geq k$, even if the flow is nonzero. This is a separate lower bound on A, not an addition to the three-array bound. The two counts need not be disjoint. All 45 identities and both partitions are checked against literal rows.

Mixed-column scope is explicit: all 180 mixed columns have exactly four T incidences and no V,h,S,g incidences. They vanish in all six actual star-center functionals, in all 15 general-flow and ν rows, and in the effective H^0/ helper $-\beta$ rows. The exact h-elimination certificates are replayed. Thus (F), (B), the center laws and the derived (T) have no omitted mixed term. This is the P-mix premise required by item 7; it does not extend the fixed 21-row dual to mixed columns.

If all nonzero zetas have source $U = \{u, v\}$ and flow vanishes, source balance gives

$$\mathrm{NT}_U := \sum_{\text{splits separating } u,v} (s(u) + s(v)) = w_U - 1.(T)$$

Zero $-\zeta$ classes elsewhere are allowed. A nonzero U-class has two together lines of cost at least 4; a nonzero-sum separating line costs at least 5. Therefore an endpoint with both has $f \geq 13$. At $f = 13$ the two together class lines are glue-free and the separating sum equals Γ.

Further stated support facts used below are: $Z = 0$ and nonzero flow imply $C \geq 6$; a single Z globally implies $C \geq 5$; each block with Z requires at least one loop column; if all Z lie in one block and $C \leq 4$, their two coefficients are equal and all flow vanishes. A single nonzero loop when $Z = 0$ is impossible. For a hypothetical matching $R = 26$, the proved matching lower bound 26 makes every support-decreasing kernel move impossible. In particular

$$(D, Z_u, Z_v, E) \to (D + a, Z_u - a, Z_v - a, E + a)(K)$$

rules out $C = 1$ and equal nonzero endpoint Z when D or E is present.

Useful charge formulas, including the helper support charge once, are

$$m = 2 : \Pi \geq 18 + d + z \text{and } \Pi \geq 12 + 5d + z;$$
$$m = 3 : \Pi \geq \text{ceil } (19.5 + d + z);$$
$$m = 4 : \Pi \geq 21 + d + z$$

For $Z = 0$ and D-degree d_u, $\pi(u) = f(u)/2 + d_u/2$ is at least max $(4 + d_u/2, 1 + 2.5d_u)$ when $d_u > 0$. The second bound improves by $1/2$ if $\Gamma(u) \neq 0$. An active vertex with $d_u = 0$ has $\pi(u) \geq 4.5$.

2. Small facts needed at the next budget. **Different prefixes.** At one attachment vertex, classes with different source pairs have just one private line in common. Its half is the union of the two pairs. Hence two such nonzero classes use at least three N-lines; if one has three N and the other two, they use at least four. The checker verifies all 540 such class pairs. Classes with the same source but different matching groups are not subject to this statement.

One Z and one L_3 in a block. The S rows force $\ell = -Z$, while the literal row identity is $2\delta = L_W - L_V$. Therefore $|\delta| = |Z|/2$, and the magnitude multiplying c_B in the projection is $|2\delta| = |Z|$. Do not conflate δ with its doubled projected coefficient. All 45 row identities and all 360 compatible Z/L_3 placements are checked. In particular this flow is nonzero. If two single-Z blocks have the same source and $C = 2$, projection first forces its right side (supported on that source) to vanish; the two distinct, nonparallel c columns then force both flows zero, a contradiction.

For clarity, the two S rows of each block have the form

$$Z_u + E + \sum_{\text{four } L_3 \in \text{one set}} = 0$$

and

$$Z_v + E + \sum_{\text{four } L_3 \in a\text{disjoint set}} = 0.$$

D and mixed columns are absent. Consequently an E-only loop block has $Z_u = Z_v = -E$(single-loop identity). A block with opposite nonzero endpoint Z cannot have only one loop: one E would force equality and one L_3 cannot satisfy both nonzero-Z rows. It therefore needs at least two loops. A Z-free block also cannot have exactly one nonzero loop, since either E or L_3 would violate an S row. Hence a Z-free nonzero-flow block costs at least two loops. These direct row consequences justify all loop counts in item 8. Zero flow alone does not imply $C = 0$; whenever $C = 0$ is used, the budget or the stated S-row argument rules out idle loops as well.

Low N-lines. The full line classification, including collisions from other classes, gives: F_4N-lines are glue-free; any F_5 line containing a specified nonzero N coefficient n has marginal -n or $-n/2$; an F_6N-line has marginal zero. These are the previously pinned analytic line controls, not a coefficient grid assumption. If all crossed inactive vertices have $f = 5$, their glues force the grid $\{0, \pm 1/2\}$. On this forced grid, a line with its sole N equal to -2 has, for $F \le 7$, only F_4 with $\sum 0$ or F_7 with $\sum \pm 1/2$. All 81 prescribed local forms are checked. A single nonzero class of value 2 has $f \ge 11$, as already proved at lower 29.

Balanced transport. When all zetas vanish, a balanced h-side pair (n,-n) changes two effective stars by $(-n, +n)$ on distinct lines of the same center, preserving V, h, center mass and H. At an active $f = 9$ with effective private support $15, n$ is positive: it cancels either $a + 1/2$ star on a one-glue line or $a + 1$ star on a three-glue line, and its partner creates the same positive star on an empty line. The checker verifies all 360 ordered h-side transfers and the nine tight scalar patterns.

Private support 18**.** In an effective $N = 0$ private layer whose center masses are all $1, P_0 \le 18$ implies H_e^0 in $\{0, 1/2, 1\}$. The line spectrum is $\{5, 7, 8, 9\}$ for a nonzero marginal and $\{0, 6, 9\}$ for a zero marginal. There are at least six nonzero marginals. Eight cost 40. Seven cost at least 35; total cost is even, and the next allowed increments are $2, 3, 4, 6, 9$, so there is no even cost at most 36. Thus there are exactly six, one per vertex, each $\sum -1/2$; extra zero-marginal lines do not change H^0. The center-mass hypothesis is essential and is used explicitly below.

3. Nonmatching and five-edge branches. For a nonmatching correction graph, $q = \#\{G : \Omega_G \ne 0\} \ge 2$. At $m \ge 3$ the general charges give $R \ge 25$. At $m = 2, \Pi \ge 20$ gives $R \ge 24$. Equality $\Pi = 19$ has either all zetas zero, giving $A \ge 6$, or one $D = 1/2$, which can cancel at most one of six arrays and gives $A \ge 5$.

At $m = 0, R \le 23$ forces effective private support 15(the private gap is 15 or at least 18), flow zero and $A \le 8$, hence $q = 2$. At $m = 1$, only $\Pi = 17$ can fit; equality gives all zetas zero, $A = 6, C = X = 0$. Its effective private layer has $P_0 \le 17$, center masses 1, hence $P_0 = 15$. In either case $H^0 = w$. By (M), the two supported Ω values sum to -2. A shared edge would therefore have weight 2, contradicting $H^0 \le 1$. If disjoint, they form a six-cycle. Each nonzero weight lies in $\{1/2, 1\}$, and the two weights sum to 2, so both are 1. Each vertex's halfmates must be its two cycle neighbors. This forces three single glues and twelve positive stars in twelve different charged blocks. For $m = 0, (B)$ forces $A \ge 12$. For $m = 1$, the balanced transport loses at most one block, so $A \ge 11$. Both contradict $A \le 8$(indeed $A = 6$ at $m = 1$). The checker verifies all 24 possible balanced transfers of this fixed six-cycle private core against V, h, H and charged-block incidences.

Now suppose the correction support has exactly five edges. The proved signed five-edge census gives an edge of weight 2. At $m \ge 3, R \ge 25$. For $m = 0, A \ge 6$; nonzero flow gives $R \ge 27$, while zero flow at $R \le 24$ gives $P_0 \le 18$, contradicting the preceding $H^0 \le 1$ lemma. At $m = 1$, a single Z gives $\Pi \ge 18, A \ge 5, C \ge 5$. Two or more Z give $\Pi \ge 20$ and per-group complement ≥ 6. Otherwise all zetas vanish, $A \ge 6$, flow is zero and $P_0 \le \Pi \le 18$ with effective center masses 1, again impossible.

For $m = 2$, $\Pi \ge 21$ already gives $R \ge 25$. At $\Pi = 20$ and $R \le 24$, the per-group charge forces $q = 2$, and both supported matchings contain $U = K$. Their four arrays outside U are nonzero, so $A \ge 4$. The budget therefore forces $A = 4$ and $C = X = 0$, hence $Z = 0$. Two D would give $\Pi \ge 22$, so there is at most one D, necessarily with source U. That D can affect the U-array in only one of the two supported groups. The other group's U-array remains nonzero, giving $A \ge 5$; with no D, all six arrays remain nonzero. Both alternatives contradict $A = 4$.

At $\Pi = 19$, all-zero zetas give $A \ge 6$. Otherwise equality is one $D = 1/2$ with glue-free endpoints and simple inactive vertices. To fit $R = 24$ its array must cancel one of six; thus $q = 2, A = 5, C = X = 0$. Cancellation gives $\Omega_D = -1$, and (M) then gives $\Omega_{\text{other}} = -1$. Write their matchings G_D, G'. Formula (F) at zero flow has $\beta = 1$ only in the D block. Take the G'-edge at an active endpoint u. If it is U, (F) requires $H^0_U = 1$, but both endpoints are glue-free. Otherwise it is outside G_D and the helper contribution is zero, so $H^0_e = 1$; one endpoint is glue-free and the other simple, giving $H^0_e \le 1/2$. Contradiction. Thus the five-edge retained support is at least 25.

4. Matching reductions and all-zero zetas. For the remainder suppose matching $R = 26$, normalized and support-minimal. Then $\Omega = -2e_M$. If all zetas vanish, $Z = D = 0$ and $A \ge 3$. With zero flow the effective private lemma gives $P_0 \ge 24$, so $R \ge 27$. With nonzero flow, $C \ge 6$ and $\Pi \ge 17$ force $\Pi = 17, A = 3, C = 6, X = 0, m = 1$. The effective private layer has center masses 1 and $P_0 \le 17$, so $P_0 = 15$. Its positive stars occupy at least six charged blocks, by the proved private 15 classification. The balanced

transport in item 2 loses at most one, leaves every actual star nonnegative, and has $\beta = 0$. Identity (B) gives $A \geq 5$, contradiction. $N = 0$ is included using private support 15 or at least 18: the first has $A \geq 6, C \geq 6$, and the second has either $P \geq 24$ at zero flow or $C \geq 6$. All give $R \geq 27$.

Thus at least one helper is nonzero. $m = 6$ gives $\Pi \geq 27$. At $m = 5$, coverage requires another direction; $\Pi = 25$ equality also forces $A \geq 3$, whereas $\Pi = 26$ has no budget for coverage. Hence $m \leq 4$.

4.1 One attachment vertex. $D = 0$. For $z \geq 3, \Pi \geq 14 + 3z, C \geq z, A \geq 2$ give $R \geq 28$.

If $z = 1$, its group must be M; otherwise $A \geq 4$ and $\Pi = 18 + C = 5$ already give 27. If $A = 2$ the Z equals 2. The single-class bound $f \geq 11$ forces $\Pi \geq 19$, so equality requires $\Pi = 19, C = 5, A = 2, X = 0$. All inactive vertices have $f = 5$ and the active one $f = 11$. Three N in the nonzero class would use three lines and cost 12. Four N, a star and a glue would exhaust $f = 11$, but the sole glue would have to equal $\Gamma = 3/2$, contradicting the crossed inactive $f = 5$ grid. Hence there are precisely two $N = -2$. Each of their distinct lines costs at least 4; since $f = 11$, each costs at most 7. The forced-grid $F \leq 7$ check therefore covers this budget. Their two lines must cost $4 + 4$ or $4 + 7$, and have total marginal 0 or $\pm 1/2$, not $3/2$. An additional nonempty glue line cannot fit. Contradiction.

If $A \geq 3$, the only remaining budget is $\Pi = 18, A = 3, C = 5, X = 0$. First, $\Gamma = 0$ cannot occur at its active $f = 9$. Two F_4N-lines leave a cost 1 remainder, which no nonempty line can carry; $\mathrm{F}_4 + \mathrm{F}_5$ has nonzero marginal; an extra balanced class costs at least four N plus one actual star, already 10. Thus $\Gamma \neq 0$. The active $f = 9$ classification forces the Z coefficient $1 : \Gamma = a - 1/2$ must equal a or $a/2$ from its F_5N-line. Its crossed inactive $f = 5$ lines force that line to be $(-1/2, +1/2, +1/2)$, not a single glue of value 1. Its positive entries force full crossing lines, activating all six vertices in one private split. The marginal totals of its two halves would then be $1/2 - 1/2 - 1/2 = -1/2$ and $-1/2 - 1/2 - 1/2 = -3/2$. They must be equal, a contradiction.

Let $z = 2$. $\Pi \geq 20, C \geq 2, A \geq 3$; at most one Z block belongs to M. The possible budgets are $\Pi = 20, A = 3($ with $C = 2$ or $3), \Pi = 20, A = 4, C = 2$, or $\Pi = 21, A = 3, C = 2$; X uses any otherwise unspent unit. In the first $A = 3$ case the M coefficient is 2. Write b for the other. The active $f = 11$ and inactive $f = 5$ force exactly four N, one actual star and one glue. Its coefficient is $\pm 1/2$, so $\Gamma = 2 + b - 1/2$ makes $b = -2$ or -1. Every N is therefore an integer. A single $\pm 1/2$ glue creates two nonzero effective half-star entries, which integer N cannot both remove; one actual star is impossible. This argument does not depend on C or X.

For $\Pi = 20, A = 4, C = 2$, both loops are single $L_3 s$, one in each Z block. Projection at the attachment vertex gives $a + b = 0$. Different source pairs would require three N-lines, costing $12 > f = 11$. Equal source pairs

instead give a zero projection and two nonparallel c columns, forcing zero flows contrary to item 2. This excludes R1c.

For $\Pi = 21, A = 3, C = 2$, the coefficients are 2 and -2 by the same projection. Equal source pairs are again impossible. With different pairs, four N use at least three lines. Five N would use at least four lines, and six N plus a star and a glue cost 15. Thus there are exactly four N, all ± 2. The active f is at least 13; since total cost is 38, it is exactly 13 and every inactive vertex has $f = 5$. Its three lines cost $5 + 4 + 4$. The only glue-bearing line contains an N of magnitude 2, so its sum is ± 2 or ± 1, not $\Gamma = -1/2$. This excludes R1b and completes $m = 1$.

5. Four attachment vertices. $\Pi \geq 23$. At $\Pi = 23$ equality permits no Z and only disjoint $D = 1/2$, with glue-free endpoints. Equality needs $A \geq 3$, hence $A = 3, C = X = 0$. Every D block must belong to M, since an outside-M array would add A. One D contradicts (T): its endpoints are glue-free. For two D on disjoint M-edges U,V, formula (F) requires $H^0_U = 1/2$; its endpoints are glue-free, so $H^0_U = 0$. This excludes R4a.

At $\Pi = 24$ with Z present, $C \geq 2$ leaves $A = 0, C = 2, X = 0$. To cancel all three M arrays, K must contain one whole M-edge U and one endpoint of each other M-edge. The two crossing M-blocks need $Z = 2$. Since $d + z \leq 3$, the remaining helper must be $D = 1$ on U; a third Z block would require $C \geq 3$. All four active Γ values are nonzero, so $f \geq 9$ at each and $\Pi \geq 5 + 18 + 3 = 26$, contradiction.

Now $Z = 0$. $A = 0$ is impossible because not every M-edge is inside K. At $\Pi = 24$ this leaves $C \leq 1$, and the S rows exclude a lone nonzero loop, so $C = 0$. $A = 1$ forces $D = 1$ on two M-blocks whose sources cover K. Their four Γ values are $1/2$, giving $\Pi \geq 25$. Thus only $A = 2, X = 0$ remains at $\Pi = 24$. At least one M array must cancel, requiring $D = 1$ on its block (M,U).

Three or more D already cost $\Pi \geq 26$: the least D-degree distribution on four active vertices is $(2, 2, 1, 1)$, which costs $6 + 6 + 4.5 + 4.5$ plus 5 for the inactive vertices. More concentrated distributions cost more. With one D, all four active vertices have $\Gamma \neq 0$. $\Pi = 24$ forces every active $f = 9$, so its two D endpoints have only together glues and $\mathrm{NT}_U = 0$, contrary to (T). With two D, an outside-M block would force an extra array; cancelling two M arrays as well would need two M D columns, exhausting $d = 2$. Therefore the second D is on another M-edge V, with coefficient a. If $a \neq 1/2$ all four Γ values are nonzero, giving $\Pi \geq 25$. If $a = 1/2, \Pi = 24$ forces $f = 9$ at $U, f = 8$ at V, and inactive $f = 5$. At U all marginal $1/2$ lies on together lines, so $H^0_U = -1$. Formula (F), with betas 2 and 1 on U,V, instead requires $H^0_U = -1/2$. This excludes R4b.

At $\Pi = 25, A = 1, C = X = 0$ is forced. There are precisely two $D = 1$ on disjoint M-edges U,V. All active $f = 9$, inactive $f = 5$. At each D endpoint the F_5 line has $N = -1, \sum 1/2$, three glues $(-1/2, +1/2, +1/2)$, and no actual star. Its F_4 line gives a positive unit star. Every actual star is

positive. The F_4 stars at the two endpoints of U have different source pairs, neither U nor V: each source contains the other endpoint of U and a third vertex. Their charged blocks therefore have $\beta = 0$ and are distinct. Identity (B) forces $A \geq 2$. The 720 literal source-incidence controls verify this claim. This excludes R4c. $\Pi = 26$ leaves no coverage budget. All $m = 4$ cases are excluded.

6. Three attachment vertices. First $Z = 0$. Nonzero flow cannot fit; $d = 0$ was excluded in item 4. For $d = 1$, (T) forces one endpoint $f \geq 13$. The other endpoint is at least 8, the third active vertex has $f \geq 9$ and the inactive cost is 15, so $\Pi \geq 24$. To fit $R = 26$ requires $A = 2, D = 1$ on an M-edge U, and $C = X = 0$. Both D endpoints then have $\Gamma = 1/2$. The tight costs are $(13, 9, 9)$ at active vertices and $(5, 5, 5)$ inactive. The separating sum is $1/2$, not $\mathrm{NT}_U = 1$.

For $d = 2$ with a common source, the charges $6+6+4.5+7.5$ give $\Pi \geq 24$; at least one outside-M array makes $A \geq 3$. For different sources, they form a path on K and $\Pi \geq 23, A \geq 3$. Equality $R = 26$ requires one $MD = 1$ on U and a second, outside-M D of coefficient a. The common endpoint has charge at least $6 + (1/2)[a + 1/2 \neq 0]$, the M-leaf at least 5, and the other leaf at least $4.5 + (1/2)[a - 1/2 \neq 0]$. Add inactive 7.5. The two indicators cannot both vanish, so $\Pi \geq 24$, contradiction. For $d \geq 3$, the degree count gives $\Pi \geq 26$, before the at least two M arrays.

For Z present, a single Z has $C \geq 5$ and $\Pi \geq 22$, impossible. Thus $z \geq 2, C \geq 2$. At $\Pi = 22$, the formula $\Pi \geq \mathrm{ceil}\,(19.5 + d + z)$ forces $d = 0, z = 2$. The total line cost 40 is just one above the minimum 39. Hence the two Z are $1/2$ at different vertices, with $f = 8$ there and $f = 9$ at the remaining active vertex. No M array cancels, so $A \geq 3$ and $R \geq 27$.

At $\Pi = 23$, put $h = d + z \leq 3$. If $A = 0, K$ must meet all three M-edges once, and cancellation requires three $Z = 2$ in three blocks. Each vertex then has $f \geq 11$, giving $\Pi \geq 27$. The remaining possibility is $A = 1, C = 2, X = 0$. If K contains an M-edge U and a third vertex w, an untouched M-edge uses the one A. Thus both the U array and the crossing M-array at w must cancel. The latter needs $Z_w = 2$. For $h = 2$, the other Z must be 2 at an endpoint of U; $f \geq 11, 11, 9$ yields $\Pi \geq 25$. For $h = 3$, all remaining helpers must be in the U block: either D plus one Z, or two Z. The costs $f(w) \geq 11$ and $f \geq 8$ at the other two vertices already give $\Pi \geq 24$. A helper elsewhere would create an additional array.

If K meets all three M-edges once, $A = 1$ requires $Z = 2$ at two vertices. For $h = 2$ their two $f \geq 11$ and the remaining $f \geq 9$ give $\Pi \geq 25$. For $h = 3$, a third Z in the remaining M-block gives $\Pi \geq 26$; a third Z elsewhere or a D creates an extra outside-M array while the remaining M-array still needs A. At $\Pi \geq 24, A = 0, C = 2$ is the only possible residual budget, but canceling all M-arrays either is impossible because an M-edge avoids K or needs three separate Z blocks and $C \geq 3$. This completes $m = 3$ by the stated cases.

7. Two attachment vertices, no Z. Let $U = K$. The flow is zero: otherwise $Z = 0$ requires $C \geq 6$, and even the minimum $\Pi = 19$ and two unavoidable M alternatings give $R \geq 27$. Three D cost $\Pi \geq 27$. With two D, each endpoint has at least four N and a star, hence $f \geq 10$. Source balance requires a separating line at an endpoint, so their f-sum is at least $13 + 10$. Add inactive 20 and two D : $\Pi \geq 24$. Their distinct blocks give $A \geq 3$, impossible. Thus there is one D, coefficient a, and flow is zero.

Source balance gives $\Pi \geq 22$. In fact $\Pi = 22$ is impossible. If $a \neq 1/2$, both Γ values are nonzero and the only endpoint costs are $13, 9$, with inactive $f = 5$. At $f = 9$, the F_5 marginal a or $a/2$ must equal $a - 1/2$, forcing $a = 1$. At $f = 13$ the separating marginal equals $\Gamma = 1/2$, whereas (T) requires $+1$ or -1.

If $a = 1/2, \Gamma = 0$ at both endpoints. A separating endpoint has $f \geq 13$, but equality 13 makes its together lines F_4 and its separating marginal equal $\Gamma = 0$, impossible. Hence $f \geq 14$. Total cost 42, inactive cost 20 and the other active cost 8 force exactly $(f_u, f_v) = (14, 8)$, with every inactive $f = 5$. The $f = 8$ endpoint is glue-free, so the $f = 14$ endpoint alone has separating $\sum S = \mathrm{NT} = \pm 1$ and together $\sum T = -S$. Its main nonzero U-class needs two distinct together lines, each cost at least 4. Three main N would use three such lines, costing 12 before a separator of cost at least 5; hence there are precisely two main $N = -1/2$. Any additional together line, including a balanced-N-only line, costs at least 4 and would give at least $4 + 4 + 4 + 5 = 17 > 14$. Thus all together marginal is in the two main lines. Their combined cost is at most 9, giving $(4, 4)$ or $(4, 5)$. The full collision-safe F_4/F_5 classification gives T in $\{0, 1/4, 1/2\}$, even with additional balanced N colliding with a main N. This contradicts $T = \pm 1$. No arbitrary balanced-class coefficient has been bounded or enumerated. Therefore $\Pi \geq 23$.

A D outside the group-M block of an M-edge would leave all three M arrays and add its own, so $A \geq 4$ and $R \geq 27$. Therefore D is on (M,U) with U in M. The two other M-arrays require $A \geq 2$. No outside-M array can carry an alternating column when $A \leq 3$: its zero total would need two such columns in addition to these two. $C = 0$. With $X = 0$ the stated 21-row dual on all private columns, all N at U, all M alternatings and this D reads $-a + t_U^M/2 = 0$. Since $t_U^M = -2 + 2a$ it gives $-1 = 0$.

The only remaining mixed budget is $\Pi = 23, A = 2, C = 0, X = 1, a = 1$. Both endpoint Γ values are $1/2, \mathrm{NT}_U = 1$, and their f-sum is at most 24. If a separating endpoint has $f = 13$, its separating marginal is $1/2$, requiring a separating line at the other endpoint as well; then both $f \geq 13$, impossible. Otherwise its f is 14 or 15 and the other is at most 10, so the first alone has separating marginal 1. Its together marginal must be $-1/2$. Its two main U-class lines cost at most 10, since a separating line costs at least 5. A third main-class N would already use three lines, costing 12 before the separator. Thus each of the two main N coefficients is -1. Additional N from balanced classes may collide in these lines; the full F_5/F_6 classification allows that. The possible two-line costs are $(4, 4), (4, 5), (5, 5), (4, 6)$; each

has nonnegative total marginal. An additional together line, including a balanced-N-only line, would cost at least 4, giving at least $8+4+5 = 17 > 15$. The required together marginal $-1/2$ is impossible.

8. Two attachment vertices, Z present. There is an M-edge avoiding K, so $A \geq 1$. Minimality gives $C \geq 2$ and coverage gives $A + C \geq 3$; consequently $\Pi \leq 23$. One Z would have $\Pi \geq 20, C \geq 5, A \geq 2$ and cannot fit. Thus $z \geq 2$. The charge formulas force $d \leq 1$ and $h = d + z \leq 5$.

For $h = 5, \Pi = 23$ is equality in $\Pi \geq 18 + h$: both active $f = 8$, inactive $f = 5$. The active Γ values vanish, so $\sum \beta = \sum \Psi = 1$. The budget forces $A = 1, C = 2, X = 0$. Its sole alternating is on the unaffected M-edge with coefficient -2, so the total law instead gives $\sum \beta = 4$. Contradiction.

For $h = 4$, equality $\Pi = 22$ would give one nonzero class per endpoint. Four Z without D use too many classes; one D fixes the classes at both endpoints, but only two Z can share that block, not three. At $\Pi = 23$ the budget is $A = 1, C = 2, X = 0$. K must meet two M-edges, and their two arrays must cancel by $Z = 2$ in their separate M-blocks. The other two helpers must cancel their β outside M. They are either a pair of opposite Z or a D plus $Z = -2D$ in a third block. Either placement requires a third Z block and $C \geq 3$. Contradiction.

For $h = 3, \Pi = 21$ again has active $f = 8$, hence one nonzero class per endpoint. Three Z are impossible. One D and two Z must share one block, with equal Z coefficients $1/2 - D$; move (K) lowers support, impossible. $A = 1$ is impossible for $h = 3$ at any Π: it would need two $Z = 2$ in the two affected M-blocks and a third helper outside them, giving an extra array. Thus the only $h = 3$ budget is $\Pi = 22, A = 2, C = 2, X = 0$.

If U belongs to M, the two other M-arrays exhaust A and $\beta_{M,U} = 2$. For $d = 0, z = 3$, one $Z = 2$ in that block leaves an opposite Z pair in another block; that pair requires at least two loops, in addition to the first block's loop. Two Z in the M-block would leave an uncancelled third Z elsewhere. For $d = 1, z = 2$, putting all helpers in the M-block invokes the one-block lemma and equal-Z kernel move. $D = 1$ in the M-block and opposite Z elsewhere also violates the one-block lemma. The only other zero-array placement is an outside-M D canceled by $Z = -2D$, and $Z = 2$ in the M-block. These are two single-Z blocks with the same source U and only two loops, excluded by item 2. Other placements leave a nonzero outside-M array.

If U is outside M, its unaffected M-edge consumes one A. For $d = 0, z = 3$, canceling both affected M-arrays needs two $Z = 2$ and leaves a third Z block, so $C \geq 3$. Canceling just one leaves two opposite Z in a block outside M; they require two loops plus the first block's loop. Canceling neither needs three M alternatings. For $d = 1, z = 2$, canceling both affected M-arrays uses $Z = 2$ at each endpoint and leaves a D array. Each endpoint then has two different nonzero classes, at least four N and a star, so $f \geq 10$ there and $\Pi \geq 10 + 10 + 3 = 23$, contradiction. Canceling neither crossing M-array

gives $A \geq 3$. Canceling just one requires aD/Z block with $\beta = 0$ outside M. At the endpoint of the M Z, its source and the D source differ, so three N-lines give $f \geq 12$; the other endpoint has $f \geq 8$. Then $\Pi \geq 23$ again.

It remains $d = 0, z = 2, \Pi \geq 20$. If both Z lie in the same block, $A \geq 2$ and $C \leq 4$. The one-block lemma gives equal Z and zero flow. Source balance needs a separating endpoint $f \geq 13$, with the other at least 8; including inactive 20 and $z = 2$ gives $\Pi \geq 23$. With $A \geq 2, C \geq 2$ this is 27.

If both distinct Z blocks are outside $M, A \geq 5$ gives $R \geq 27$. If both are in M, the projection's right side has opposite nonzero values on the affected M-edge endpoints, while every M-block c is constant on each M-edge. A non-M flow block is therefore necessary, costing two loops in addition to the two Z blocks: $C \geq 4$. With $A \geq 3$ this is 27. For $A = 2$ at least one Z is 2, giving $\Pi \geq 22$; for $A = 1$ both are 2, giving $\Pi \geq 23$. These are even larger totals.

Finally suppose one Z block is in M and one is outside. $A \geq 3$. At $A = 3$ the M Z equals 2; if they attach at different vertices, $f \geq 11 + 8$; if at the same vertex, four N and a star cost at least 10 there and the other active zero $-\zeta$ vertex costs 9. Either way $\Pi \geq 22$, so $R \geq 27$. The only remaining budget is $\Pi = 20, A = 4, C = 2, X = 0$, with one $Z = 1/2$ at each glue-free endpoint, inactive $f = 5$ and one L_3 in each Z block. Let e be the source of the M Z. At its active endpoint $B = 0$; its other endpoint is either also active or simple inactive. Hence $H_e^0 \leq 1/2$. The first helper contributes $1/2$ to (F). The second contributes $1/2$ if e is its source, $1/4$ if e is its side, or 0 otherwise. If e is its source, both flows have zero divergence there, while (F) is at most $1/2 + 1 - 2 = -1/2$. If it is a side, the required divergence is at most $1/2 + 3/4 - 2 = -3/4$, while the sole possible contributing flow has magnitude $1/4$. If neither, the required divergence is at most -1 and the actual divergence is zero. Contradiction in every case. This excludes R2c, R2d and R2e, including both placements U in M and U outside M. The checker verifies all 112 prescribed rational corner controls.

Thus every matching identity has retained $R \geq 27$.

Global conclusion. Matching corrections require $R \geq 27$; five corrections require $R \geq 25$; at least six require $R \geq 24$. Therefore

$$W = e + R \geq \min\{3 + 27, 5 + 25, 6 + 24\} = 30.$$

Together with the correction census, this excludes every width-29 boundary and proves $W^*(6, 2) \geq 30$.

S13.5. **The lower bound 31.**
Scope. The preceding bound is $W \geq 30$; suppose $W = 30$.
Obstruction. Part A lists the budgets, Part C removes nonmatching corrections, and Parts D and F–I remove matching $R = 27$.
Conclusion. $W \geq 31$, without a retained-29 exclusion.

A. Complete outer cover and noncircular minimality. Suppose a PM2 representation uses at most 30 directions. The preceding global bound $W \geq 30$

forces its width to be exactly 30. Canonical reduction and normalization preserve the identity and never increase support, so their outputs also have width 30. Consequently, an exact kernel move that strictly decreases support would contradict the preceding bound. This is the minimality principle used throughout the section; it does not rely on the bound $W \geq 31$ that we are proving.

Write $W = e+R$. The canonical correction graph has nonzero signed edge weights w of coordinate degree 2 after the standard scale 3 conversion. No vertex is isolated. Fewer than 3 edges cannot cover 6 vertices; 3 edges give a perfect matching, with each weight 2. Four edges are impossible. For a graph on six nonisolated vertices with four edges, its number of components c satisfies $c \geq 6 - 4 = 2$ and $c \leq 3$. If $c = 3$, all components have size 2 and the simple graph has only three edges. Thus $c = 2$, and four edges attain the minimum $6 - c$, so the graph is a forest, of sizes $2 + 4$ or $3 + 3$. The degree equations force equal cardinalities of the two bipartition classes in each component, ruling out odd components and the 4-star; the remaining 4-path would have a zero middle edge. This is valid for signed weights.

The preceding retained-support bounds give $R \geq 27$ when $e = 3$ is matching, $R \geq 25$ when $e = 5$, and $R \geq 24$ for every other nonmatching graph. Thus $e \geq 7$ already costs at least 31. Exactly the following budgets need exclusion:

Correction support	Retained support at width 30	Component
$e = 5$	$R = 25$	Part C: nonmatching
$e = 6$	$R = 24$	Part C: nonmatching
$e = 3$, matching	$R = 27$	Parts D–I: matching

For matching $R = 27$, first ask whether every normalized endpoint class ζ vanishes. Part F treats that case for every attachment count, using the private-cost lemma in Part E. Otherwise a nonzero class needs nonzero N coefficients on both h-sides at its vertex. Every surviving D or Z therefore attaches in K, and at least one attachment is present.

Normalization changes only D, Z, E. It preserves every class value ζ, every N coefficient, the attachment set K, the correction support e, and the correction weights w, while never increasing R. The remaining proof separates the nonzero-helper cases by $m = |K|$: Part G treats one attachment, Part H two, Part I three, and Part D four through six. Balanced N classes with $\zeta = 0$ remain allowed at every counted attachment.

B. Inputs, notation, and scope. The canonical model covers all $22,155$ primitive projective rays of the PM2 dictionary and reduces them, without increasing support, to 1530 retained directions and 15 correction directions.

We use that reduction, its certificates, and the preceding global lower bound $W \geq 30$ as established inputs.

We use the support decomposition $R = \Pi + A + C + X$. The private-and-helper cost Π counts stars, glues, and columns of types N, D, Z; the loop cost is $C = L_3 + E$. Let K be the set of N-attachment vertices, $m = |K|$, and $h = d + z$. All these costs count nonzero columns, not sums of coefficients.

On a private line, write $F = g + 2s + 2n$, where g, s, n count its glues, actual stars, and N columns. Let $f(x)$ be the sum of line costs at center x. Then

$$2\Pi = \sum_x f(x) + 2h.$$

An inactive center has $f \geq 5$ and cannot have $f = 6$. An active center has $f \geq 8$, strengthened to $f \geq 9$ when $\Gamma \neq 0$. Actual star coefficients sum to 1 at each center. These facts allow signed stars, balanced N classes, and collisions. Costs are charged to actual selected columns, so a star replaced by an N contribution is not charged twice.

The D-degree bound used in Parts D and I is the $Z = 0$ bound in S13.4, item 1. For incident D-degree $d_x > 0$, the charge $\pi(x) = f(x)/2 + d_x/2$ is at least $\max(4 + d_x/2, 1 + 5d_x/2)$; the second bound increases by $1/2$ if $\Gamma(x) \neq 0$. An active vertex with $d_x = 0$ has $\pi(x) \geq 9/2$. The single-class value 2 bound $f \geq 11$ is recorded in S13.4, item 2 "Low N-lines"; its $f = 9$ equality law is in S13.4, item 1 and the single-Z application in S13.4, item 4.1. These allow balanced N. Part H, item 3.1 also proves the needed single-class $f \leq 10$ values directly. Part I uses the weaker bound $f \geq 10$ when another nonzero class is present.

For $B = (G, U = \{u, v\})$ define $\zeta_u = Z_u + D, \zeta_v = Z_v + D$ and $\beta = Z_u + Z_v + 2D$. Normalization removes D if an endpoint ζ vanishes; Z attaches only in K. Set $\Gamma(x) = \sum_B \zeta_x(B) - 1/2$. A nonzero class needs two N on distinct lines, one on each h-side.

The following equations hold for the full retained system: mixed columns contribute zero to each of them. Thus using these equations does not assume $X = 0$.

$$t_U^G = \Omega_G + \beta_{G,U},$$

$$\text{div } (\delta)_e = H_e^0 + \sum_B \beta_B(1[e = U] + 1[e = V]/2 + 1[e = W]/2) - w_e,$$

$$H_{uv}^0 = (B_{uv} + B_{vu})/2,$$

$$B_{uv} = -2 \sum_{\substack{\text{halves at} u \\ \text{containing } v}} \text{line}_{\text{ marginal}},$$

$$2\sum_B \delta_B(1_W - 1_V) = \sum_B (Z_v - Z_u)(e_u - e_v),$$

$$\text{starSum } (B) + \text{alternatingSum } (B) = \beta_B$$

Orient flow from V to W. Its divergence is $-\delta$ at V and $+\delta$ at W. The loop identity is $2\delta = L_W - L_V$; a single Z and its sole L_3 give $|\delta| = |Z|/2$. The group-array partition and block-alternating partition are different. Their counts are not added without a checked disjointness claim. Mixed columns vanish in all the equations used above.

The proved source identity at zero flow, when every nonzero ζ has source U, is $\mathrm{NT}_U = w_U - 1$. Balanced classes elsewhere remain allowed. The kernel move is $(D, Z_u, Z_v, E) \to (D + a, Z_u - a, Z_v - a, E + a)$. Z-free nonzero flow costs at least 6 loops. The fixed 21-row dual is used only with $C = X = 0$ and its exact 403-column scope, never as a mixed-column dual.

The modest private 15 lemma used in F is proved from the private line equations: P15 gives positive stars and $A \geq 6$. A three-glue private line forces all six active lines into one split, whose stars have distinct block sources; otherwise there are three single-glue generators, twelve positive stars, and at most two stars per block. The incidence check covers all ten private splits and all 216 generator configurations.

The lettered parts below have their own numbered items; each reference identifies both its section and its part.

C. Nonmatching $e = 5$ and $e = 6$ exclusions.

1. Inputs, budgets, and scope. Use the proved scale with center star masses 1, signed correction weights w, $\sum w = 6$, and $\sum \Omega = -2$. Write $\alpha_G = -\Omega_G$; then $\sum \alpha = 2$. Coefficients remain arbitrary signed rationals.

The proved canonical census excludes $e < 3$ and $e = 4$; $e = 3$ is a perfect matching. The complete possible width 30 budgets under the current branch bounds are:

e	$R = 30 - e$	Proved retained lower bound	Status
3	27	27	Matching branch; handled in the matching cases
4	26	—	Correction census excludes $e = 4$
5	25	25	Nonmatching boundary analyzed below
6	24	24	Nonmatching boundary analyzed below
≥ 7	≤ 23	24	Already excluded

The proved inputs used here are canonical completeness, the correction census, normalization, the general V- and h-equations, array equations, and flow equations, signed total mass, center mass, $f \geq 5$ at inactive and $f \geq 8$ at active vertices ($f \geq 9$ when $\Gamma \neq 0$), source balance, the $Z = 0$ nonzero-flow cost $C \geq 6$, and the single-Z cost $C \geq 5$. These are general inputs; neither the matching $\Omega = -2e_M$ reduction nor the 37-direction obstruction for the fixed construction is used.

Notation: $R = \Pi + A + C + X$, $\Pi = \text{private } + h$, $h = d + z$, private $=$ stars $+$ glues $+ N$, $m = |K|$, and $q = \#\{G : \Omega_G \neq 0\}$. Nonmatching corrections have $q \geq 2$.

2. Each helper can cancel at most one array. **Lemma H.** Every normalized identity satisfies

$$R \geq \left\lceil 15 + \frac{3m}{2} \right\rceil + 3q + C + X. \qquad \text{(H)}$$

Proof. Before helper cancellation, the q supported matching groups have $3q$ arrays with nonzero Ω. The exact array equations are $t_U^G = \Omega_G + \beta_{G,U}$. Each D or Z belongs to one β block and therefore can help cancel at most one of these arrays. Multiple helpers sharing a block still cancel at most that one array. At most h arrays are canceled, so $A \geq 3q - h$. Extra nonzero arrays outside those groups only strengthen this bound. Array supports are disjoint; zero-total arrays are allowed to contain arbitrarily many canceling columns.

Meanwhile

$$2(\Pi - h) = \sum f(u) \geq 5(6 - m) + 8m,$$
$$\Pi \geq \left\lceil 15 + \frac{3m}{2} \right\rceil + h$$

Adding the two bounds cancels the helper charge h and proves (H). No sign assumption or addition of the block-star and array counts is made. For $q \geq 2$, the $m = 2, 3, 4$ lower bounds are respectively $24, 26, 27$ before $C + X$. Thus the width 30 nonmatching boundaries have $m \leq 2$. At $m = 2, R \leq 25$ also forces $q = 2$.

If $m = 1$ and $z \geq 2$, each Z is a different nonzero attachment class and block. Each needs two N, and there is an actual star. Therefore

$$\Pi \geq 14 + 3z, A \geq 3q - z, C \geq z,$$
$$R \geq 14 + 3q + 3z \geq 26$$

If $m = 1$ and $z = 1$, inequality (H) and the lower bound $C \geq 5$ give $R \geq 17 + 6 + 5 = 28$. Thus, at either nonmatching boundary, every normalized class ζ vanishes when $m = 0$ or $m = 1$.

3. Effective private support at most nineteen. **Lemma P19.** In an effective $N = 0$ private layer with all center masses 1, private support $P_0 \le 19$ implies

$$H_e^0 \le \frac{3}{2} \qquad \text{for every edge } e. \qquad \text{(P19)}$$

H^0 is not asserted to be nonnegative.

Proof. On a three-position private line with glue vector l and marginal $r = \sum(l)$, the effective stars are $l - r \cdot 1$ and the cost is $F = g + 2 \cdot \text{star}_{\text{count}}$. The complete signed spectrum is

$$r \neq 0 : F \in \{5, 7, 8, 9\}; r = 0 : F \in \{0, 6, 9\}$$

For one glue, F_5 is automatic. For two nonzero glues a, b, the effective stars are $(-b, -a, -a - b)$: F_6 when $a + b = 0$ and F_8 otherwise. Three glues give $F_5, 7, 9$ at nonzero marginal, and F_9 at zero marginal. At F_5, every nonzero glue has magnitude $|r|$: the forms are one glue of coefficient r, or three glues $(-r, r, r)$ up to permutation.

Center mass 1 gives total marginal $-1/2$ at each vertex, so there are at least six nonzero marginal lines. Total line cost is $2P_0 \le 38$.

- With exactly six nonzero marginals, each vertex has one, of value $-1/2$. Zero marginals contribute nothing to H^0. Every directed B row has two entries 1 and the rest 0; hence $H^0 \le 1$.
- Eight nonzero marginals would cost 40. Seven cost at least 35; the allowed positive increments are $2, 3, 4$ and zero-line increments $6, 9$. The only even total at most 38 is 38**: six** F_5 **and one** F_8, with no nonempty zero-marginal line.

In the second case one vertex u has two distinct private lines; the other five have one each. The two u-lines belong to different private partitions. A glue meets the opposite line in the same partition. Therefore at least one F_5 line at u meets only ordinary F_5 lines of other centers: if F_8 is at u, its other line does; if F_8 is at another center, at most one of the two u-lines shares its partition. Those ordinary lines have marginal $-1/2$ and every nonzero glue of magnitude $1/2$. The chosen u-line therefore has marginal $\pm 1/2$. Its companion is nonzero and their sum is $-1/2$, forcing the two marginals to be $+1/2$ **and** -1.

Thus the duplicated center has directed entries

$$B_{uv} = -2 \cdot \sum(\text{together marginals })$$

in $\{0, -1, 2, 1\}$. Every other center has B entries 0 or 1. Since $H_{uv}^0 = (B_{uv} + B_{vu})/2$, every edge is at most $3/2$. The argument allows arbitrary signed coefficients; it does not restrict them to a grid.

When all zetas vanish, effective elimination creates at most one star position per N column, so $P_0 \le \text{stars } + \text{glues } + N = \Pi$. Its center masses

remain 1. At zero flow and $\beta = 0$, the general equation gives $H^0 = w$. Consequently **P19 excludes any boundary with every $\zeta = 0$ and zero flow with $\Pi \leq 19$ and a correction edge of weight** 2.

4. Signed graph facts; no global $q = 2$ assumption. For $e = 5$, the proved census supplies an edge of weight 2. One can also classify its signed degree-law graphs directly:

- K_2 plus C_4, with K_2 weight 2 and alternating cycle weights $a, 2 - a$, both nonzero.
- The connected double-star, with four leaf edges weight 2 and its central edge weight -2.

Reason: every vertex has nonzero graph degree. A leaf edge has weight 2, so its neighbor cannot have graph degree 2. A connected six-vertex, five-edge graph is a tree. The leaf-neighbor rule and degree sum 10 leave only degree sequence $(3, 3, 1, 1, 1, 1)$; the two degree 3 vertices join and each has two leaves, giving the double-star weights. The other tree degree sequences either have a degree 2 leaf-neighbor or are the star, whose center equation fails. In a disconnected graph, no three-vertex component can fit $e = 5$: its only degree-law possibility is a triangle, and two triangles need six edges. The only remaining split is $2 + 4$. A connected four-vertex four-edge component is C_4 or triangle-with-tail; the latter's degree equations force a triangle edge zero. This leaves $K_2 + C_4$.

An explicit signed control is $01 = -2, 02 = 03 = 14 = 15 = 2$. Every weighted degree is 2 and the signed edge sum is 6. This control is **not a full PM2 identity**. It shows why $e = 5$ must not globally be assumed to be a pair of overlapping matchings. That pair description is used below only after the retained budget forces $q = 2$.

For $q = 2$, two distinct matchings either share one edge (giving $e = 5$ or are disjoint (giving $e = 6$, since $\alpha_1 + \alpha_2 = 2$ and both α are nonzero. For $q = 3$, each matching has a unique edge not in the other two: each other matching can share at most one of its three edges. If all correction weights equal 1, each of these unique edges forces $\alpha_G = 1$, contradicting $\sum \alpha = 2$. This will exclude the low-budget $e = 6, q = 3$ alternative.

5. Five correction edges require $R \geq 26$. Suppose $e = 5$ and $R = 25$, and choose a normalized representative. Its total width is 30, so Part A supplies support minimality. Inequality (H) excludes $m \geq 3$.

For $m = 0, m = 1$, Part C, item 2 excludes any Z. All zetas vanish. Since $A \geq 6$, $\Pi \leq 19$. Nonzero flow would require $C \geq 6$, giving at least 27 at $m = 0$ or 29 at $m = 1$. Thus flow is zero. P19 and $P_0 \leq \Pi$ contradict the weight 2 edge in the five-edge census. This covers arbitrary mixed coefficients as well: the center equations, the V- and h-equations, and the general flow and source-balance equations have no X term.

For $m = 2$, inequality (H) forces $q = 2$. If a physical Z is present, then $C \geq 2$. To justify this, suppose there were only one loop. Every Z would

lie in its block. If the loop were an L_3 column, exactly one Z would remain, contradicting the sole-Z bound $C \geq 5$. If it were an E column, the endpoint Z coefficients would be equal, and the exact kernel move on (D, Z_u, Z_v, E) would decrease support. This contradicts the width-30 minimality principle. Therefore (H) gives $R \geq 24 + C \geq 26$ whenever $Z > 0$.

Let $Z = 0$. Nonzero flow cannot fit because it would require $C \geq 6$. If $d = 0$, all zetas are zero and again $A = 6, \Pi \leq 19$ contradict P19. If $d \geq 2$, each endpoint has at least $2d$ columns of type N and an actual star, while the four inactive centers cost at least 10 private support. Thus

$$\Pi \geq 12 + 5d, A \geq 6 - d, R \geq 18 + 4d \geq 26$$

It remains to consider one D, with coefficient a, on source K. Here $A \geq 5$ and $19 \leq \Pi \leq 20$. Its group must be one of the two supported Ω groups; otherwise all six supported arrays remain and its outside array adds another, giving $A \geq 7$. Call that group G_1, with α_1, and the other G_2, with α_2. The matchings share a correction edge e_0 of weight 2.

If $K = e_0$, source balance gives $\mathrm{NT}_K = 1$. A nonzero separator costs at least 13 at one endpoint and at least 8 at the other. Adding inactive cost 20 and D gives $\Pi \geq 22$, impossible.

If K is a unique G_1 edge, its weight is α_1. The same source balance gives $\mathrm{NT}_K = \alpha_1 - 1$; a nonzero value would again cost $\Pi \geq 22$. Hence $\alpha_1 = \alpha_2 = 1$. If $\Pi = 19, a \neq 1/2$ would make both Γ values nonzero, forcing active costs $9 + 9$ and $\Pi \geq 20$. Thus $a = 1/2$. If $\Pi = 20$, the remaining budget permits only $A = 5$; one of the six arrays must cancel, which requires $2a = \alpha_1 = 1$. Again $a = 1/2$.

Every inactive cost f is at most 7: total line cost is at most 38, the active centers cost at least 16, and the other three inactive centers cost at least 15. An inactive $f \leq 7$ has one nonzero marginal line of value $-1/2$; any second such line costs at least 5, and any nonempty zero line costs 6. Its B row therefore has two entries 1 and the rest 0.

The shared edge e_0 is disjoint from K, so both its endpoints are inactive and $H^0_{e_0} \leq 1$. The D helper is on G_1 with e_0 as a side; $\beta = 1$ supplies only $1/2$ there. Zero-flow (F) consequently requires

$$H^0_{e_0} = w_{e_0} - 1/2 = 3/2,$$

a contradiction. All $e = 5, R = 25$ cases, including the possible one-mixed $\Pi = 19$ boundary, are excluded.

6. The six-cycle private lemma and a signed erasure charge. **Lemma C18 (the six-cycle core).** Suppose an effective private layer has $N = 0$, center mass 1 at every vertex, and $P_0 \leq 18$. If H^0 equal to the unit six-cycle, it is exactly the private core with three glues $-1/2$ and twelve stars $+1/2$. Its twelve stars occupy twelve distinct block-star charge blocks.

Proof. The private line count at $P_0 \leq 18$ leaves exactly six nonzero marginals, one $-1/2$ at each center. The cycle edges of weight 1 require both directed B entries 1, so each center's halfmates are its two cycle neighbors. For cycle 0123450 the selected halves are $\{i, i-1, i+1\}$. Opposite centers i and $i+3$ select complementary halves. Thus there are precisely three selected private partitions, with one nonzero row and one nonzero column in each, both of marginal $-1/2$.

A nonempty zero-marginal line costs at least 6 and has at least two glues. It touches two opposite lines; at most one of these has nonzero marginal, so it forces another nonempty zero-marginal line. Their cost at least 12 added to the six nonzero lines' cost 30 exceeds total cost 36. Therefore all zero-marginal lines are empty. Each selected private block contains only the glue at the intersection of its selected row and column, with coefficient $-1/2$, yielding four $+1/2$ stars. The fixed cycle core is checked directly against the V- and h-equations, center masses, all fifteen H rows, and the literal block-star ownership; the twelve charged blocks are distinct. Relabeling covers every six-cycle. This core is a forced private relaxation, not an asserted full identity.

Lemma B (unchanged blocks). With $\beta = 0$, if the effective stars have b nonzero block-star sums, then actual alternating support satisfies $A \geq b - N_{\text{count}}$. Each N contributes to one V-star position and hence at most one such block. At least $b - N_{\text{count}}$ block sums are unchanged and nonzero. The literal B equation forces an alternating column in each; their ownership sets are disjoint. Changed stars can have arbitrary signs. This count is not added to any separate array lower bound.

7. Six correction edges require $R \geq 25$. Assume $e = 6, R = 24$. By $(H), m \geq 3$ is impossible.

For $m = 0, m = 1, Z$ is excluded by Part C, item 2. Nonzero flow would require $C \geq 6$, whereas $\Pi \geq 15$ and $A \geq 6$ leave at most three loop terms; hence flow is zero. Since $A \geq 6$, the effective private layer has center masses 1 and $P_0 \leq \Pi \leq 18$. The proved private 18 lemma gives H^0 in $\{0, 1/2, 1\}$. There are six nonzero correction weights and their signed sum is 6. Thus every weight is 1.

At $m = 1, q = 2$ is forced by $\Pi \geq 17$. At $m = 0, q \leq 3$; $q = 3$ is excluded by the unique-edge argument in Part C, item 4. Thus in both cases $q = 2$ consists of disjoint matchings of weights 1, and the graph is a six-cycle. Apply Lemma C18 from Part C, item 6.

- At $m = 0$, actual and effective stars agree and force $A \geq 12$, contradicting $R = 24$ and private support 15.
- At $m = 1$, total line cost is at most 36; five inactive vertices cost 25, so the active $f \leq 11$. $\Gamma = -1/2$ requires at least one glue and actual center mass 1 requires at least one star. Hence there are at most four N. Lemma B gives $A \geq 12 - 4 = 8$, while $\Pi \geq 17$ permits $A \leq 7$. This treats arbitrary

balanced coefficients and possible N collisions, without assuming actual stars are positive.

For $m = 2, (H)$ forces $q = 2$ and $C = X = 0$. Hence $Z = 0$. With $d = 0, A = 6$ plus $\Pi \geq 19$ already gives 25. With $d \geq 2$ the $18 + 4d$ bound above gives at least 26. With $d = 1$, exactly $\Pi = 19, A = 5$ remains. Its D group must be supported and its array must cancel. The $\Pi = 19$ cost forces $a = 1/2$ and glue-free active endpoints, with simple $f = 5$ inactive vertices. Therefore $\alpha_1 = 2a = 1$ and $\alpha_2 = 1$. $e = 6$ means the two matchings are disjoint. At an active endpoint u of the D source, its G_2-edge e lies outside G_1, so receives no helper. Its other endpoint is inactive. Thus $H_e^0 \leq 1/2$ but zero-flow (F) requires $H_e^0 = w_e = 1$, a contradiction. All $e = 6, R = 24$ cases are excluded.

D. Matching rigidity and the $m = 4, m = 5, m = 6$ exclusions.

1. Helper–array count and stronger matching rigidity. Let $h = d + z$ count helper directions, let q be the support size of Ω, and let m count the N-attachment vertices. Keep all notation and normalization from the lower bound 30 proof. Exactly $3q$ group arrays have nonzero Ω. At most h blocks can have nonzero β: each D or Z belongs to just one block. A supported-group array can vanish only in such a block. Every other supported-group array has a nonzero total and needs an alternating column. Therefore

$$A \geq 3q - h$$

Meanwhile $2\Pi = \sum f + 2h \geq 5(6 - m) + 8m + 2h$, giving

$$\Pi \geq 15 + \left\lceil \frac{3m}{2} \right\rceil + h,$$

$$R \geq 15 + \left\lceil \frac{3m}{2} \right\rceil + 3q + C + X. \qquad \text{(HA)}$$

The argument is signed, uses the group-array partition only, and does not double-count a helper shared with an N class. A helper may fail to cancel any array, but cannot cancel two. A block containing several helpers still cancels at most one array.

For matching corrections, a nonzero $\Omega + 2e_M$ relation has at least six matching-incidence terms, by the proved signed kernel lemma. Thus any Ω other than $-2e_M$ has $q \geq 5$. Formula (HA) then forces $R \geq 30$, even at $m = 0$. Consequently

$$R \leq 29 \text{ with matching corrections} \quad \Longrightarrow \quad \Omega = -2e_M. \qquad \text{(O29)}$$

This extends the scope of the prior $R \leq 26$ rigidity without assuming that the old statement already covered $R = 27$. For the remainder assume matching $R = 27$ and use (O29).

The proved matching lower 27 makes every support-decreasing normalized kernel move impossible at $R = 27$. Thus $C = 1$ remains impossible when Z is present; with $Z = 0$ the S rows also rule out one nonzero loop.

2. Two small facts. **No $f = 10$ at a single value 1 helper class.** Suppose the vertex has one nonzero ζ class of value 1, no other nonzero helper class, and $\Gamma = 1/2$. Then $f = 10$ is impossible. Three N in the main class use three lines and cost 12. Any extra balanced class adds at least two N; four N, one actual star and one glue cost 11. Thus $f \le 10$ leaves the main pair $N = -1, -1$ on distinct lines and no other N. Additional nonempty lines cannot fit. The only exact cost 10 pairs are $\mathrm{F}_4 + \mathrm{F}_6$ or $\mathrm{F}_5 + \mathrm{F}_5$. Their total marginals are respectively 0 or one of $1, 3/2, 2$ by the full collision-safe $\mathrm{F}_4/\mathrm{F}_5/\mathrm{F}_6$ lemma; none equals $1/2$.

Two unit D columns cannot fit $\Pi \le 26$ and $A \le 2$. Suppose the only helpers are $D = 1$ in (M, U) and (M, V), where U, V are disjoint matching edges and $K = U \cup V$. Assume $C = 0$ and $\Pi \le 26$. The third matching array has total -2, whereas the other two and all outside-M arrays have total 0. An outside-M alternating array would need two columns in addition to the nonzero M-array, so $A \le 2$ makes every alternating column a group-M column.

The block-alternating columns in each of the three group-M loop blocks are disjoint from all twelve group-M alternating columns. This is an explicit incidence fact, checked on all 45 block–group combinations; it is not an identification of the two alternating partitions. Hence the two D blocks have no block-alternating coefficient. The physical-block identity (B) requires the actual star coefficients to sum to 2 in each of the two D blocks. Their combined star mass must therefore be 4.

But each active endpoint has $\Gamma = 1/2$ and $f \ge 9$, and each inactive vertex $f \ge 5$. Including $d = 2, \Pi \ge 25$. The line-cost budget up to $\Pi = 26$ has only two units of slack above active $(9, 9, 9, 9)$ and inactive $(5, 5)$. An active $f = 10$ is impossible by the preceding fact. If an inactive vertex had $f = 7$, all other vertices would be at their minimum: every active $f = 9$ glue is $\pm 1/2$, and the other inactive $f = 5$ glues are in $\{0, \pm 1/2\}$. Every glue at the $f = 7$ vertex crosses one of those lines, so it too is on this grid. There is just one nonempty line at that vertex; its sum is $-1/2$. On the forced grid such a line cannot have cost 7, because the F_7 form with two stars requires a glue coefficient outside this grid. Thus the inactive vertices both have $f = 5$. Every active cost f is at most 11.

Each active endpoint has two main class lines, costing at least 4 each. A third main N line, or any other nonempty private line, would cost at least 12. Therefore all actual star support at an active vertex lies in those two main lines. Their halves contain its D source pair, so their star source pairs contain its mate and exclude the center. Such a source is neither U nor V. This is checked for every V position on every possible main line, not just the N position.

Only the two inactive centers can therefore contribute stars in the two D blocks. Their $f = 5$ patterns have nonnegative stars, and each has total actual star mass 1. Their combined contribution to any subset of blocks is at most 2, contradicting the required 4. Actual stars at a possible active $f = 11$ vertex need not be nonnegative; they are excluded from the two blocks by their source incidences. This proves the claim.

3. Matching $R = 27$ excludes $m \geq 5$. At $m = 6, \Pi \geq 27$. Equality is the stated private equality case with $Z = 0$ and disjoint $D = 1/2$; all three M-arrays remain nonzero, so $A \geq 3$. Larger Π already exceeds $R = 27$.

At $m = 5, \Pi = 25$ equality similarly requires $A \geq 3$. $\Pi = 27$ leaves no coverage direction. At $\Pi = 26$ only one direction remains outside Π. A Z would force $C \geq 2$, so $Z = 0$. The M-edge containing the inactive vertex cannot carry D, and its unchanged array has total -2. Hence $A \geq 1$, forcing $A = 1, C = X = 0$. Canceling two M-arrays requires two $D = 1$ on two M-edges inside K. No extra D can occur: an outside-M block would create another array, and the remaining M-edge has an inactive endpoint. Their four endpoints have $f \geq 9$; the fifth active vertex, whose class values ζ all vanish, also has $f \geq 9$; the inactive vertex has $f \geq 5$. Including $d = 2$ gives $\Pi \geq (5 \cdot 9 + 5)/2 + 2 = 27$, contradiction.

4. Matching $R = 27, m = 4$, nonzero helpers. The **nonzero-helper hypothesis is essential** in this section. The case $m = 4$ with every $\zeta = 0$ is covered by Part F. Now $\Pi \geq 23$. $\Pi = 27$ leaves no coverage direction.

$\Pi = 23$. Equality has $Z = 0$ and disjoint $D = 1/2$. $A \geq 3$ leaves $C \leq 1$, so $C = 0$; X may be 0 or 1. There are one or two D. One contradicts source balance at its glue-free endpoints, whether its block is in M or outside. Two D blocks outside group M would give $A \geq 5$, exceeding the budget 4. Thus with two D at least one is in M, on source U. At U, the flow equation (F) requires $H_U^0 = 2 - 1 - \eta$, where the other helper contributes $\eta \in \{0, 1/2\}$. Thus H_U^0 must be positive, whereas its glue-free endpoints give $H_U^0 = 0$. These contradictions permit $A = 4$ or the possible mixed column; neither case is silently discarded.

$\Pi = 24$ with Z. $C \geq 2$ leaves $A \leq 1$. Here $h \leq 3$. If $A = 0$, all three M-arrays must cancel, so K meets every M-edge and consists of one whole edge and two crossing endpoints. Exactly three helpers are available, one per canceled array. The two crossing Z must be 2; the third helper is unit D or Z of coefficient 2 on the whole edge. Every active Γ is nonzero, so $f \geq 9$ at all four vertices and $\Pi \geq 5 + 18 + 3 = 26$.

For $A = 1, C = 2$, exactly two M-arrays must cancel. A block outside M with $\beta = 0$ would need at least two helpers in addition to at least one in each canceled block, exceeding $h = 3$. A block outside M with $\beta \neq 0$ would need another alternating. Thus all helpers lie in M-blocks. $C = 2$ permits at most two Z blocks. If the whole M-edge U is canceled by unit D, a crossing M-edge e_2 by $Z_x = 2$, and the third crossing M-edge e_3 carries $Z_y \neq 2$, the active costs are at least $9, 9, 11, 8$. With inactive cost 10 and

$h = 3, \Pi \geq \lceil 47/2 \rceil + 3 = 27 > 24$. Otherwise, a Z in the third M-block forces both canceled blocks to contain Z, and a third Z block would need $C \geq 3$. A D in the third, uncanceled M-block is possible only when the two canceled arrays are the crossing ones; then those two single Z are 2 and cost $f \geq 11$ each; the two D endpoints cost at least 8 each. Including $h = 3$ and inactive cost 10 gives $\Pi \geq 27$. Thus after this exclusion all helpers lie in the two canceled blocks.

Here a single Z globally would cost $C \geq 5$, so $z \geq 2$. If $h = 2$, the two helpers are Z of coefficient 2 at distinct active endpoints. The other two active endpoints have zero ζ, giving costs at least $11, 11, 9, 9$ and $\Pi \geq 27$. If $h = 3$, one possibility is a Z pair totaling 2 in one canceled block and unit D in the other. The one-block lemma forces that pair to be $1, 1$. All four Γ are nonzero, so $\Pi \geq 26$. Otherwise a single Z of coefficient 2 is in one canceled block and a two-helper combination in the other. The Z of coefficient 2 endpoint costs 11; an active vertex outside the other helper source costs at least 9; the two remaining vertices cost at least 8 each. Thus $\Pi \geq 5 + 18 + 3 = 26$. No $\Pi = 24$ placement survives.

$\Pi = 24, Z = 0$**.** $A \geq 1$ and the budget gives $C \leq 2$. If $C = 2$ then $A = 1$, but $A = 1$ requires two unit D with $\Pi \geq 25$; so $C = 0$. One D has source balance and costs $\Pi \geq 26$: if $a \neq 1/2$ the endpoint costs are at least $13, 9$, and if $a = 1/2$ they are at least $14, 8$; the other active costs are $9, 9$ and inactive costs $5, 5$. Three or more D satisfy $\Pi \geq 26$ by the proved D-degree bound restated in Part B. Two D with the same source or overlapping sources also cost at least 26 or 25 respectively; only disjoint sources can fit $\Pi = 24$.

At most one of those D blocks can be outside M. If one is outside, $A \leq 3$ requires cancellation by unit D in the M block. The other D must be $1/2$ to fit $\Pi = 24$; otherwise all four active Γ are nonzero and $\Pi \geq 25$. The M endpoints have $f = 9$, so $H^0_U = -1$. Formula (F) requires $H^0_U = 0$ or $-1/2$ according as U is absent from, or a side of, the other helper group. Contradiction.

If both D are in M, either both coefficients are $1/2$, or exactly one differs from $1/2$. If both differ, $\Pi \geq 25$. With both $1/2$, each endpoint has $f \leq 10$; every nonzero-marginal line must be a together line because a separator would cost at least 13. Thus $H^0_U = 0$, whereas (F) needs $1/2$. With just one different coefficient a, tight cost forces its endpoints $f = 9$, the other endpoints $f = 8$ and inactive $f = 5$. The F_5 marginal equation $a - 1/2$ in $\{a, a/2\}$ forces $a = 1$. Now $H^0_U = -1$ whereas (F) needs $-1/2$. This excludes $\Pi = 24$, including $X = 1$ when $A = 2$.

$\Pi = 25$ **with** Z**.** Only $A = 0, C = 2$ can fit. Now $h \leq 4$. Again all three M-arrays must cancel and K contains one M-edge and two crossing endpoints. Helpers outside those M-blocks cannot occur: they would need at least two helpers for $\beta = 0$, beyond the three required cancellations. The crossing Z are 2. At $h = 3$ the whole-edge helper is unit D or Z of coefficient 2, giving $\Pi \geq 26$ as above. At $h = 4$ the two whole-edge endpoints cost at

least 8 each and the crossing endpoints at least 11 each; including $h = 4$ gives $\Pi \geq 5 + 19 + 4 = 28$. Contradiction.

$\Pi = 25, Z = 0$**.** $A \geq 1$ makes $C \leq 1$, hence $C = 0$. If $A = 1$, two unit D in M are forced. The two-D lemma in Part D, item 2 excludes $A = 1$ and also $A = 2$ with those coefficients. For $A = 2$, at least one M array cancels, so there is a unit D in (M, U). One D costs $\Pi \geq 26$ and three cost $\Pi \geq 26$. For two D, a common source costs $\Pi \geq 26$. If their different sources overlap, the second source is not an M-edge, since the first is U in M. Its D therefore has an outside-M group and a nonzero outside array. The two unchanged M-arrays plus that outside array force $A \geq 3 > 2$. Disjoint sources remain. A second D outside group M would give two unchanged M-arrays plus its own, exceeding $A = 2$. Thus the second D lies in (M, V), with coefficient b.

If b is neither $1/2$ nor 1, all four Γ are nonzero and $\Pi = 25$ forces all active $f = 9$; the F_5 equation would force $b = 1$, contradiction. If $b = 1/2$, every endpoint has $f \leq 11$, below the separator cost 13, so $H^0_U = -1$ still; (F) needs $-1/2$. If $b = 1$, Part D, item 2 excludes it. Thus $\Pi = 25$ is impossible.

$\Pi = 26$**.** Only $A = 1, C = X = 0$ can cover. Z is impossible. Canceling two M-arrays forces precisely two unit D in M: any additional D has a nonzero array outside M, or would use an inactive endpoint in the remaining M-edge. The two-D lemma excludes $\Pi = 26, A = 1$. This finishes every $m = 4$ case with nonzero helpers.

E. Private support at least 25 *under* $H^0 = 2M$*.*

Statement and stated definitions. Every vertex has total center mass one. For each private 3|3 partition σ and vertex u, let $a_{\sigma,u}$ be its center mass on that line. The directed marginal matrix is

$$B_{uv} = \sum_{\sigma\text{: } u,v \text{ lie in the same half}} a_{\sigma,u}$$

Assume $H_{uv} = (B_{uv} + B_{vu})/2$ equals 2 on the edges of a perfect matching M and zero on every other pair. Then the private support satisfies $P_0 \geq 25$.

The stated private line spectrum is as follows. A nonzero-marginal line has cost in $\{5, 7, 8, 9\}$; a nonempty zero-marginal line has cost 6 or 9. The sum of line costs is $2P_0$. A simple vertex has exactly one nonzero center-mass line; its mass is one and its B row has exactly two entries one, all other entries zero. These facts are proved in Section S13.1, Appendix A and the uniform matching reduction.

Proof. Suppose $P_0 \leq 24$. No matching edge can have two simple endpoints: its H value would belong to $\{0, 1/2, 1\}$, rather than equal two. At least three vertices are therefore nonsimple, and at least nine lines have nonzero marginal. Ten such lines already cost at least $50 > 2P_0$. Thus there are exactly nine: precisely three vertices are simple, and the other three have exactly two nonzero-marginal lines each. Every matching edge joins one simple vertex to one two-line vertex.

Let v be a two-line vertex and s its simple matching partner. The two line masses at v sum to one. Since B_{sv} is zero or one, the equation $B_{vs}+B_{sv}=4$ forces B_{vs} to be 4 or 3. The partner s must consequently belong to exactly one of the two half triples at v: belonging to both gives $B_{vs}=1$, and belonging to neither gives zero. Write the two masses as p and $1-p$, where $p\in\{3,4\}$. For any other vertex y, its membership in those two halves gives

$$B_{vy}\in\{0,1,p,1-p\}$$

In particular, **no entry in the row of any two-line vertex is** -1. No simple vertex has an entry -1 either.

Now fix a simple vertex s. Its unique half has two other vertices, so its row has two entries $B_{sy}=1$. At most one of those vertices is the matching partner of s. For the other one, $H_{sy}=0$ requires $B_{ys}=-1$. The preceding paragraph excludes this for every possible y. Contradiction.

Therefore P_0 cannot be at most 24, and $P_0\geq 25$.

The proof never assumes that a nonzero center mass is positive. It also allows nonempty zero-marginal lines: they affect support cost but not B, and cannot change the nine-line argument.

F. All normalized zetas zero, every attachment count. Use the stated notation $R=\Pi+A+C+X$, effective private support $P_0\leq\Pi$, and the rigidity (O29) of Part D, item 1. When all zetas vanish, normalized $D=Z=0,\beta=0$, and $A\geq 3$. All effective center masses equal one. With $Z=0$ every nonzero-flow block costs at least two loop columns, and a nonzero flow needs at least three blocks.

1. Zero flow. Part E gives $H^0=2M\Longrightarrow P_0\geq 25$. Thus $R\geq P_0+A\geq 28$. This excludes zero flow for every number of N attachment vertices, including the previously retained $\Pi=P_0=24, A=3$ boundary.

2. A three-edge flow forces the six-cycle core. Suppose $P_0\leq 18$ and a nonzero flow has at most three supported blocks. The private-support lemma for $P_0\leq 18$ gives H^0 entries in $\{0,1/2,1\}$. Moreover $Z=0$ makes $\sum_B \delta_B\cdot c_B=0$ by projection. Summing the edge divergences over pairs containing a coordinate x gives

$$\sum_{e\ni x}\operatorname{div}(\delta)_e=\sum_B \delta_B c_B(x)=0.$$

Since $\beta=0$, the flow equation (F) therefore yields

$$\sum_{e\ni x}H^0_e=\sum_{e\ni x}w_e=2.$$

Thus H^0 has coordinate degree two and its edge weights sum to six. The flow divergence is H^0-2M.

Each of the three matching pair-nodes has strictly negative demand. Their total negative demand is $6-\sum_{e\in M}H^0_e\geq 3$. All other demands

are nonnegative and at most one, so at least three other pair-nodes have positive demand. Three flow edges can touch at most six demanded nodes. Therefore there are exactly six, and the flow consists of three disjoint edges, each joining a positive demand to a negative demand. This uses signed flows correctly: every connected component has zero total divergence, regardless of edge orientation.

There are exactly three positive demands, each at most one. Their sum is at least three, so all equal one. It follows that H^0 equals one on all three matching edges and on exactly three further edges, and is zero elsewhere. Its support is a simple 2-regular graph on six vertices. The only possibilities are a six-cycle or two triangles. The latter contains no perfect matching, whereas the support contains M. Thus H^0 is a six-cycle and M is one of its two alternating perfect matchings.

The effective private layer is then the stated fixed six-cycle core: three glues of coefficient $-1/2$ and twelve stars of coefficient $+1/2$, in twelve distinct star blocks. For completeness, the private-support lemma for $P_0 \le 18$ forces exactly one nonzero-marginal line per vertex, with marginal $-1/2$. Its halfmates must be the two cycle neighbors. Opposite cycle vertices therefore give the two nonzero-marginal lines in each of three splits. The available excess over their baseline cost 30 is at most six, so there is at most one nonempty zero-marginal line. Such a line requires two nonzero crossing cells, but the opposite half has only one other nonempty line; no second zero-marginal line can fit. This is impossible. With all other lines empty, each used split has just one glue between its opposite cycle vertices. Hence $P_0 = 15$ and the stated twelve stars are forced. Their distinct literal block incidences are already pinned and are independently reconstructed by the literal-incidence checker.

3. No N directions. Here $\Pi = P_0 = P$. Lemma SL3 gives the private support gap $P = 15$ or $P \ge 18$. At $P = 15$, the private-support lemma gives $A \ge 6$. Nonzero flow gives $C \ge 6$, so $R = 27$ forces $A = 6, C = 6, X = 0$ and at most three flow blocks. Part F, item 2 then gives the six-cycle core. Since no N is present, its twelve blocks have positive actual star sums and $\beta = 0$, so (B) gives $A \ge 12 > 6$, contradiction. This uses only the modest private 15 incidence lemma restated in Part B. For $P \ge 18, R = 27, A \ge 3$ and $C \ge 6$ force $P = 18, A = 3, C = 6, X = 0$. There are at most three nonzero-flow blocks, so Part F, item 2 forces $P = 15$, contradiction.

4. At least one N direction. The stated charge gives $\Pi \ge 17$. From $R = 27, A \ge 3, C \ge 6$ we obtain $\Pi \le 18$ and $C \le 7$. Thus the nonzero flow again has exactly three blocks. Part F, item 2 gives the effective six-cycle core $P_0 = 15$.

The charge $\Pi \ge 15 + 2m$ confines N to a single attachment vertex u. The five inactive vertices cost at least five each, so $f(u) \le 2\Pi - 25 \le 11$. $\Gamma(u) = -1/2$ requires at least one glue, and the actual center row requires at least one actual star. Consequently

$$1 + 2 + 2N \le f(u) \le 11, \text{hence } N \le 4$$

Each N direction changes exactly one effective star cell, and every star cell belongs to one block. At most four of the twelve positive core blocks can change. At least eight therefore retain their original actual star sum $1/2$. Since $\beta = 0$, identity (B) requires at least one alternating column in each of these eight disjoint blocks. Thus $A \ge 8$. The budget instead gives $A \le 27 - 17 - 6 = 4$, contradiction.

No sign condition is imposed on stars in changed blocks. N collisions can only reduce the number of affected blocks. The argument does not add the block-alternating count to the group-array count.

G. One attachment vertex and useful $m = 2, m = 3$ reductions. We use the helper-array bound and matching rigidity for $R \le 29$, proved in Part D, together with the general equations used for the preceding lower bound $W \ge 30$. Write $K = \{u\}$. Normalization excludes D, since a surviving D requires two nonzero attachment endpoints. Every Z attaches at u, in a distinct block and a distinct nonzero class. The only block in group M available at u is $(M, \{u, u^*\})$. Other Z blocks lie outside M. Hence $A \ge 2$ always, and for $z = 2$ one has $A \ge 3$. If $z = 2$ and $A = 3$, exactly one Z block is in M and its coefficient is 2.

1. A single Z requires at least six loop directions. **Lemma L1$^+$ (the sole-Z loop bound).** In any normalized identity with exactly one nonzero Z, the number C of loop directions is at least six. D may be arbitrary.

Let the Z have coefficient $z \ne 0$ at u in $B = (U = \{u, t\}, V, W)$. Divide the projected flow equation by its nonzero coefficient z. Its right side is $\varrho = e_t - e_u$. A block flow contributes $q_B c_B$, where $c_B = 1_W - 1_V$ and $q_B = 2\delta_B/z$. The vector c_B vanishes exactly on its source U.

If the Z block contains one loop direction, its S rows force that loop to be an L_3, and the proved exact scale gives $q_B = +1$ or -1. The residual $\varrho - q_B c_B$ has six nonzero coordinates, each $+1$ or -1. One other block cannot supply it. If two other blocks supplied it, their four-element supports would have to cover all six coordinates, with intersection size two and two exclusive coordinates apiece. The exclusive coordinates force both flow coefficients to have absolute value one. At a shared coordinate their sum is then 0 or ± 2, never ± 1. Thus at least three further blocks are necessary. All are Z-free and need at least two loop directions each, giving $C \ge 1 + 3 \cdot 2 = 7$.

If the Z block has at least two loop directions, its residual has support two when $q_B = 0$ and support six when $q_B \ne 0$. It cannot equal one nonzero four-supported block vector. At least two other nonzero-flow blocks are therefore required. They are Z-free, so $C \ge 2 + 2 \cdot 2 = 6$. This proves L1$^+$.

This is a signed argument. D never appears in the projected equation or the relevant S rows.

2. Two Z blocks with at most three loop directions. Suppose the two Z columns attach at u, have coefficients a, b, and have sources $\{u, t_1\}, \{u, t_2\}$. If $C \le 3$, every flow block must be one of these two: an additional Z-free block costs two loops besides the required one in each Z block. The projected equation at coordinate u therefore gives $a + b = 0$, since both block vectors vanish at u. In particular $\Gamma(u) = -1/2$.

If the sources coincide, the projected right side is zero. The two distinct block vectors are nonparallel, so both flows vanish. Each single-Z block then needs at least two loops: one L_3 alone has nonzero flow by the exact scale used in Part G, item 1. Thus $C \ge 4$, contradiction. Consequently, whenever $C \le 3$ the sources are different and $b = -a$.

The two different-source N classes have at most one common private line. Their common half, when used, is exactly $\{u, t_1, t_2\}$. Each nonzero class needs at least two N on distinct lines. Therefore four N need at least three lines. Five N need at least four lines: they must be split as three in one class and two in the other, since any additional balanced class needs at least two N. These are the proved class-incidence facts.

If one Z block is in M, relabel $t_1 = u^*$, with coefficient a. At both source edges $\{u, u^*\}$ and $\{u, t_2\}$, the actual flow divergence is zero: neither edge can be a side of either block because it contains u. Neither helper contributes to the other one's source edge for the same reason. Equation (F) consequently gives the exact pair of identities

$$H^0_{u,u^*} = 2 - a, \qquad H^0_{u,t_2} = a. \qquad \text{(E)}$$

3. Low-cost common line with two opposite N coefficients. The following collision-aware local fact will be used at $\Pi = 22$. Suppose a line contains exactly two N columns, with coefficients n and $-n$, where $n \neq 0$, and has total cost at most seven. Then its possibilities are:

- At cost F_4, there is no glue or actual star, and the two N columns occupy the same cell.
- At cost F_6, the marginal is zero.
- At cost F_7, the marginal is n or $-n$.

Proof. Let v be the aggregate N vector. Its entries sum to zero, and the two N contributions may occupy the same cell. With glue vector l and marginal $s = \sum(l)$, the actual star vector is $l - s \cdot 1 - v$. After the four units charged to the two N, at most three remain for glues plus twice the actual-star count.

With no actual stars, summing the star equation gives $s = 0$, hence $l = v$. This gives F_4 if the N collide, and F_6 otherwise. With an actual star, there is exactly one star and at most one glue. A nonzero glue and the two opposite N can leave exactly one star only when the N occupy the two nonglue positions, the glue coefficient is $+n$ or $-n$, and the actual star coefficient is twice its negative. This is F_7 with $s = \pm n$. No other case fits the cost.

The accompanying checker verifies these affine equations after the legitimate homogeneous normalization $n = 1$; it does not impose a coefficient grid.

For a line with one of the main N, the proved full $F_4/F_5/F_6$ lemma gives F_4 glue-free, F_5 marginal $-n$ or $-n/2$, and F_6 marginal zero. This includes the collision-safe scope of the proved lemma.

4. Complete helper and support budgets. If $z \geq 3$, each helper needs two N. The five inactive vertices contribute at least 25 to $\sum_x f(x)$, and u has at least one actual star. Thus $\Pi \geq 14 + 3z$. Each Z block needs a loop, $C \geq z$, and $A \geq 2$, giving $R \geq 28$. It remains $z = 1$ or $z = 2$.

4.1 One Z. Here $\Pi \geq 18$ and $C \geq 6$ by $L1^+$. A Z in a block outside group M needs $A \geq 4$, already giving $R \geq 28$. If $A = 2$, the M array cancels, so $Z = 2$ and the proved single-class bound gives $\Pi \geq 19$. The only retained-support budgets with $R = 27$ are therefore

$$(\Pi, A, C, X) = (19, 2, 6, 0), \text{with } Z = 2;$$
$$(\Pi, A, C, X) = (18, 3, 6, 0)$$

The first has active $f = 11$ and inactive $f = 5$. The preceding argument for a sole Z of coefficient 2 at an active center of cost $f = 11$ excludes it: crossed $f = 5$ lines force the half-grid; there are precisely two main $N = -2$, and their possible low-cost line marginals cannot sum to $\Gamma = 3/2$. The second has active $f = 9$ and inactive $f = 5$. The proved $f = 9$ argument forces $Z = 1$ and a positive dense line whose crossing lines give unequal marginal totals on the two halves of its private split. These are the two single-Z arguments in S13.4, item 4.1: the paragraph starting "If $z = 1$" and the following paragraph starting "If $A \geq 3$". They use only active $f = 11$ or $f = 9$, inactive $f = 5$, the V- and h-equations, center equations, and crossing-glue magnitudes. The arguments do not depend on C, X, or flow; the arguments therefore apply unchanged to these $C = 6$ budgets.

4.2 Two Z: exhaustive tuples. Now $\Pi \geq 20, A \geq 3$ and $C \geq 2$. The two Z are at the same endpoint and therefore occupy distinct blocks, each of which needs a loop. Distributing the two remaining directions gives exactly these ten tuples:

Π	A	C	X	Exclusion below
20	3	2	2	I
20	3	3	1	I
20	3	4	0	I
20	4	2	1	II
20	5	2	0	II
20	4	3	0	II

Π	A	C	X	Exclusion below
21	3	3	0	III
21	3	2	1	III
21	4	2	0	III
22	3	2	0	IV

I. $\Pi = 20, A = 3$**, every allowed allocation to** C **and** X**.** The Z in group M has coefficient 2. At least four N and one actual star give active cost at least 10. The total line cost is 36; inactive costs are 5 or at least 7. Thus every inactive vertex has $f = 5$ and the active cost is exactly 11. Its odd cost requires a glue, making the costs tight: exactly four N, one star and one glue. The glue is $\pm 1/2$. Writing b for the other Z coefficient, we have $\Gamma = 2 + b - 1/2 \in \{\pm 1/2\}$, so $b = -2$ or -1. Every N coefficient is now an integer. One half-integral glue creates two nonzero half-integral effective star entries. Integer N cannot remove either, contradicting the single actual star. This is the proved half-integrality argument and imposes no condition on C or X.

II. $\Pi = 20, A \geq 4, C \leq 3$**.** Part G, item 2 forces different sources and $b = -a$. The four required N use at least three lines, each costing at least four. That contradicts active $f \leq 11$. Equal sources were already ruled out by their zero-flow $C \geq 4$ cost.

III. $\Pi = 21, C \leq 3$**.** Again the sources differ and $b = -a$. At least three N-lines are needed; $\Gamma = -1/2$ rules out all three being F_4. Thus $f(u) \geq 13$. The total line budget is 38, so active $f = 13$ and inactive $f = 5$ are forced. There are exactly four N: five would occupy four lines costing at least 16, and six plus an actual star and glue already cost at least 15.

The two classes therefore use exactly three lines, with their overlap containing opposite N. The other two lines cost at least four each, so the common line costs at most five. Part G, item 3 makes it the glue-free F_4 collision. The other two lines must be F_4 and F_5. Hence the active vertex has a unique nonzero marginal line, of marginal $-1/2$; this line is not the common class line. Every B row has only entries zero or one, so both H^0 values in (E) lie in $[0, 1]$. $A \leq 4$ ensures one Z block is in M, since two outside blocks would need $A \geq 5$. Applying both equations (E) forces $a = 1$ and both H^0 values to equal one. The active half must then contain both u^* and t_2. It is exactly the common class half $\{u, u^*, t_2\}$, contradicting that the common line is glue-free.

IV. $\Pi = 22, A = 3, C = 2$**.** The Z in group M has coefficient $a = 2$, the other is $b = -2$, and the sources differ. The total line budget is 40. As above active $f \geq 13$, so active $f \leq 15$ and every inactive $f \leq 7$. An inactive $f \leq 7$ has exactly one nonzero-marginal line with marginal $-1/2$. Its B row

therefore has only zeros and ones, even if its individual glues are not on the half-grid.

There are at least four N and at most six, since seven N plus a glue and an actual star cost at least 17. Five N would use four lines and cost at least 16, so N is four or six.

If $N = 6$, its twelve units leave only one glue and one actual star within $f \leq 15$. The active vertex has a single nonzero-marginal line with marginal $-1/2$, so its B row is also in $\{0, 1\}$. Equation (E) at the non-M source would require $H^0 = 2$, whereas $H^0 \leq 1$. Contradiction. This covers the extra balanced N class that must not be silently excluded from the $f = 15$ boundary.

If $N = 4$, there are exactly three N-lines; a fourth nonempty line would cost at least five beyond their baseline 12. Two N of opposite values $2, -2$ occupy the common line, and the other two lines each have one N. The common line has cost at most seven. If it is F_7, its marginal is ± 2, and the other lines must both be F_4; they cannot sum to $-1/2$. If it is F_6, its marginal is zero and the other two lines cost at most nine, leaving $\mathrm{F}_4 + \mathrm{F}_4$ or $\mathrm{F}_4 + \mathrm{F}_5$. $\mathrm{F}_4 + \mathrm{F}_4$ gives total marginal 0, and $\mathrm{F}_4 + \mathrm{F}_5$ gives a marginal in $\{\pm 1, \pm 2\}$. Neither equals $-1/2$. If the common line is F_4, the two remaining lines have combined cost at most eleven. Apart from $\mathrm{F}_4 + \mathrm{F}_7$, every possible pair drawn from $\mathrm{F}_4/\mathrm{F}_5/\mathrm{F}_6$ has an integer marginal sum, not $-1/2$. Thus only $\mathrm{F}_4 + \mathrm{F}_7$ survives the local sum condition, and the F_7 is the unique nonzero-marginal line at u. Once again its B row is in $\{0, 1\}$, and (E) requires the impossible $H^0_{u,t_2} = 2$. This finishes IV.

5. Immediate reductions of the remaining $m = 2, m = 3$ cases. **Exactly one Z is impossible at $m = 2$ or $m = 3$, even with D present.** At $m = 2, \Pi \geq 20$: if $d \geq 1$ this follows from $\Pi \geq 18 + d + z$; if $d = 0$, the active vertex without a helper has $f \geq 9$ and the other has $f \geq 8$, giving $\Pi \geq \lceil (20 + 9 + 8)/2 \rceil + 1 = 20$. Also $A \geq 2$. If K is an M-edge, the other two M-arrays are untouched. If K is not an M-edge, every D is on source K outside M, and the single Z can cancel at most one M-array. Lemma $\mathrm{L1}^+$ therefore gives $R \geq 20 + 2 + 6 = 28$.

At $m = 3, \Pi \geq 22$ with one Z. If $d \geq 1$ the generic bound gives $\Pi \geq 20 + d + z \geq 22$. If $d = 0$, two active vertices have $f \geq 9$, the Z endpoint has $f \geq 8$, and the inactive cost is at least 15, so $\Pi \geq \lceil (15 + 9 + 9 + 8)/2 \rceil + 1 = 22$. Then $\Pi + C \geq 28$ already, by $\mathrm{L1}^+$.

With $Z = 0$, nonzero flow is impossible at $m = 2$ or $m = 3$ and $R = 27$. At $m = 3$, at most one M-edge lies inside K, so at least two unchanged M-arrays need $A \geq 2$; $\Pi \geq 21$ and $C \geq 6$ already give $R \geq 29$. At $m = 2, A \geq 2$ and $\Pi \geq 19$. If $\Pi \geq 20, R \geq 28$. In $\Pi = 19$ equality, either all helpers vanish ($A \geq 3$) or the proved equality classification gives one $D = 1/2$, which cannot cancel any M-array (again $A \geq 3$). Thus $R \geq 19 + 3 + 6 = 28$.

These are preliminary reductions, not complete $m = 2, m = 3$ exclusions. The remaining possibilities there have either $Z = 0$ and zero flow, or at least two Z. Arbitrary idle loops are not excluded unless their support budget or the S equations exclude them separately.

H. Two attachment vertices.

1. Inputs and notation. The inputs are the preceding lower bound $W \geq 30$ and the local laws already stated. For clarity, we restate the helper-array argument before applying it to two attachments. Use the established scale-3 normalization, in which actual star coefficients sum to 1 at every center.

$$R = \Pi + A + C + X,$$
$$\Pi = \text{stars } + \text{glues } + N + D + Z, C = L_3 + E,$$
$$h = d + z, 2\Pi = \sum f(u) + 2h$$

For a private line, $F = g + 2s + 2n$; $f(u)$ is the sum over the lines at center u. Inactive vertices have $f \geq 5$ and first strict excess ≥ 2. Active vertices have $f \geq 8$, or $f \geq 9$ if $\Gamma \neq 0$. For $K = \{u, v\}$, all D have source K. Normalization removes a D column if either endpoint class ζ is zero.

For block $B = (G, U)$, with $U = \{a, b\}$,

$$\zeta_a = D + Z_a, \zeta_b = D + Z_b, \beta = 2D + Z_a + Z_b,$$
$$\Gamma(a) = \Psi(a) - 1/2, \Psi = \sum \zeta_a$$

A nonzero class (G, U, a) has four possible N columns, on four distinct lines whose halves contain U. The coefficients on each of its two h-sides sum to $-\zeta$. Thus, if the class is realized by exactly two N columns, one lies on each side and both have coefficient $-\zeta$. A nonempty balanced class needs at least two N columns.

The proved equations used are:

$$t_U^G = \Omega_G + \beta_{G,U},$$
$$\text{div } (\delta)_e = H_e^0 + \sum_B \beta_B \cdot (1[e = U] + 1[e = V]/2 + 1[e = W]/2) - w_e, (F)$$
$$H_{ab}^0 = (B_{ab} + B_{ba})/2,$$
$$B_{ab} = -2 \cdot \sum_{\text{halves at} a \text{containing } b} \text{line }_{\text{marginal}},$$
$$2 \cdot \sum_B \delta_B \cdot c_B = \sum_B (Z_b - Z_a) \cdot (e_a - e_b), (P)$$
$$c_B = 1_W - 1_V,$$
$$\text{starSum } (B) + \text{alternatingSum } (B) = \beta_B.(B)$$

Orient every block $V \to W$: its edge divergence is $-\delta$ at V, $+\delta$ at W, and zero at U. Thus $|\delta|$, not $|2\delta|$, is the side-edge magnitude. The literal identity is $2\delta = L_W - L_V$. A sole Z and sole compatible L_3 give $L_3 = -Z$ and $|\delta| = |Z|/2$. The checker pins all 45 orientations and literal scales, all 360 such Z/L_3 controls, and all 15 effective F rows.

Mixed-column contributions vanish in the V- and h-equations, (F), (P), (B), and the center-mass equations. The fixed 21-row dual is used **only at** $C = X = 0$. Lemma SL8 proves its extension explicitly. Its complete scope is all 270 private columns, all 120 columns of type N attached in K, all 12 group-M alternatings, and the single column $D(M, K)$, for a total of 403 columns. The dual combines homogeneous, correction-free kernel rows of the reduced model. The fifteen correction columns and the target do not enter these rows. Its equation is

$$-D(M, K) + t_K^M/2 = 0. \text{(D21)}$$

For a single D in group M, we have $t_K^M = -2 + 2D$, so (D21) gives $-1 = 0$. Relabeling covers all matchings and matching source pairs.

Other proved inputs: the signed matching-incidence kernel needs at least six supported terms; matching retained support is at least 27; one global Z costs $C \geq 5$; $Z = 0$ with nonzero flow costs $C \geq 6$; each Z block needs a loop; a Z-free nonzero-flow block needs at least two loops. At $R = 27$, a support-decreasing exact kernel move contradicts matching lower 27:

$$(D, \mathrm{Za}, \mathrm{Zb}, E) \to (D + t, \mathrm{Za} - t, \mathrm{Zb} - t, E + t). (K)$$

In particular $C = 1$ is impossible. With $Z = 0$ this also follows directly from S rows. When all nonzero zetas have source K and flow is zero, the proved source balance, allowing balanced N elsewhere, is

$$\mathrm{NT}_K = \sum_{\text{separating lines at} u,v} \text{marginal } = w_K - 1 \in \{1, -1\}. (T)$$

2. Matching rigidity at $R = 27$. Let q be the support size of Ω. Its $3q$ supported-group arrays each have nonzero Ω. A helper can change only its own array, so

$$A \geq 3q - h, \Pi \geq \left\lceil 15 + \frac{3m}{2} \right\rceil + h,$$
$$R \geq \left\lceil 15 + \frac{3m}{2} \right\rceil + 3q + C + X$$

Unless $\Omega = -2e_M$, the signed relation $\Omega + 2e_M$ is nonzero, has support at least 6, and gives $q \geq 5$. The last inequality then gives $R \geq 30$ even at $m = 0$. Thus $R = 27$ implies $\Omega = -2e_M$. Every M-array has total $\beta - 2$; every other array has total β. Nonzero array totals need separate alternating

columns. A zero-total array with nonzero alternating support needs at least two columns.

Because some M-edge avoids K, at least one M-array remains unchanged: $A \geq 1$ for the entire $m = 2$ argument.

3. Collision-safe local facts.

3.1 Low-cost lines and one-class vertices. The full line lemma allows N columns from other classes. It says that a line containing N is glue-free at cost F_4, and has marginal zero at cost F_6. At cost F_5, a specified nonzero coefficient $N = n$ forces the marginal to be $-n$ or $-n/2$. More generally, every nonzero glue on an F_5 line of nonzero marginal r has magnitude $|r|$. This last statement also holds with no N: its form is one glue or three signed equal-magnitude glues. The checker independently verifies the small exact scalar masks with one or two N and arbitrary rational second coefficient; no grid is imposed. For the marginal bound alone, any F_6 line with three N columns would have $g = s = 0$ because $2n = 6$, and would therefore have marginal 0. No per-position incidence assertion is needed for this statement.

Suppose a vertex has just one nonzero class, with value a, and $f \leq 10$. If it has four or more N columns, the bound $f \leq 10$ forces exactly four N, one actual star and no glue, so $\Gamma = 0$ and $a = 1/2$. Three main-class N use three distinct lines and cost at least 12; additional balanced classes require two N. Otherwise there are exactly two main $N = -a$ and no other N. There are no additional nonempty lines. The possibilities are:

Main line costs	Class value a forced by $\Gamma = a - 1/2$
$4 + 4$	$1/2$
$4 + 5$	1
$4 + 6$	$1/2$, both marginals 0
$5 + 5$	$-1/2$ or -1

For $5 + 5$, the marginal on each line is a or $a/2$; solving gives the stated negative values. Consequently **positive** $a > 1$ **requires** $f \geq 11$. For a center u with $f(u) \leq 10$, and any vertex v outside its class prefix, $B_{uv} \leq 2$: only one possible main half can contain v, and its negative marginal is at worst -1. The $\Gamma = 0$ cases with extra balanced N are glue-free and give $B = 0$. At $f = 9$ with nonzero Γ the class value is 1, all glue is on its F_5 line, and $\Gamma = 1/2$.

One nonzero source class plus a nonzero-marginal separating line costs at least $4 + 4 + 5 = 13$. At $f = 13$ the together F_4 lines have marginal 0, so the separating sum equals Γ. A separating line with any nonzero glue also costs at least 5, even if its marginal is 0; with two main lines this still forces $f \geq 13$. These bounds permit balanced collisions on all lines.

3.2 The sharp local bound for two nonzero classes. **Lemma** S**. Two nonzero classes at one attachment imply** $f \geq 13$**.**

If $f \leq 12$, the center-mass requirement of a nonzero actual star leaves at most five N. Six N plus a star already cost 14. Five N consist of a three-N class and a two-N class. Different prefixes have just one shared line, so their union has at least four N-lines, costing at least 16. For the same prefix but different groups, the four possible lines agree and their V positions differ on every line (90 exact incidence pairs). Five N have at least three lines; the tight distribution $(2, 2, 1)$ costs at least $5 + 5 + 4 = 14$, since two nonzero N at different V positions cannot have F_4.

Thus exactly four N remain, two per class, with coefficient pairs $(-a, -a)$ and $(-b, -b)$, and no other N. For different prefixes there are at least three lines. To fit 12 they must all be F_4. The shared line must have both N at the same V position with $a + b = 0$; the other two glue-free stars sum to $a + b = 0$, contradicting center mass 1.

For the same prefix, three lines already cost at least $5 + 4 + 4 = 13$, so both classes must use exactly the same two lines. Their N occupy different V positions. A shared F_5 line has no stars and forces $a = b$; a shared F_6 line has no stars and forces $a = -b$. A shared F_7 line cannot occur when $a = b$. These statements follow directly from $S_i = l_i - r - \nu_i$, and are independently checked by exact affine support masks. The only pairs within cost 12 are $(5, 5), (5, 6), (5, 7), (6, 6)$. The middle pairs are inconsistent and the remaining pairs have no actual star. Again center mass 1 fails.

The bound is tight locally. The checker gives two value $1/2$ classes with same prefix 01 at center 0, one F_5 shared line and two F_4 lines, $f = 13$ and center mass 1. It checks every V- and h-equation at that center. This local control is not a full PM2 identity.

3.3 Inactive lightness and the limits of star counting. An inactive vertex with $f \leq 9$ has exactly one nonzero-marginal line, whose sum is $-1/2$. Two such lines cost at least 10; an extra nonempty zero line costs at least 6 in addition to 5. Its B entries therefore belong to $\{0, 1\}$. All its actual stars have source equal to its half minus the center. This statement concerns marginals, and does not assume nonnegative stars.

The actual stars at an active center require more care: balanced classes can remove stars without increasing the local cost. The following local configuration demonstrates the issue, using the original dictionary identifiers. At center 0, take main class $(M, 01, 0)$, coefficient $D_{1437} = 1$, and main coefficients $N_{1400} = N_{1419} = -1$. Glues with identifiers $(453, 454, 457)$ and coefficients $(-1/2, 1/2, 1/2)$ produce actual stars with identifiers $(1157, 1103, 1132)$ and coefficients $(1, 1, -1)$, at cost $f = 13$.

Now add the balanced pair $N_{1397} = 1$, $N_{460} = -1$ on the same h-side. It removes stars 1157 and 1132, leaving only star 1103 with coefficient 1, while the cost remains $f = 13$. All V- and h-equations at center 0, and its center-mass equation, hold before and after; the added class has $\zeta = 0$.

Thus a unit-class configuration with a separator need not retain two positive main stars. This configuration checks a local claim only: it is neither a full identity nor a width-27 counterexample. The arguments below count literal source incidences and do not charge the two removable stars.

4. Reduce all helper placements. If $d \geq 3$, the proved $m = 2$ charge $\Pi \geq 12 + 5d + z$ gives $\Pi \geq 27$, and $A \geq 1$ excludes $R = 27$. If $d = 2$, both vertices have two distinct nonzero D classes; Lemma S gives $\Pi \geq 25 + z$. With $z = 0$ at most one D is in M, so $A \geq 3$ and $R \geq 28$. If $z = 1$, then $C \geq 5$; if $z \geq 2$, then $C \geq 2$. Each alternative exceeds $R = 27$. Thus $d \leq 1$.

If $z = 1, d = 0$, the other active vertex has zero ζ and $f \geq 9$. Thus $\Pi \geq 20, A \geq 2$ and $C \geq 5$. $A \geq 3$ already gives 28. $A = 2$ forces the Z in M with value 2; its $f \geq 11$ makes $\Pi \geq 21$ and again gives 28.

If $z = 1$ and $d = 1$, then $\Pi \geq 20$, $A \geq 2$, and $C \geq 5$. $R = 27$ requires $\Pi = 20, A = 2, C = 5, X = 0$. The equality $A = 2$ forces both helpers into block (M, K), with $\beta = 2$: an outside D or Z would add an outside array; if K is not an M-edge, D is outside and no $A = 2$ placement is possible. Private equality $\Pi = 20, h = 2$ makes both active $f = 8$, all inactive $f = 5$, hence both endpoint zetas $1/2$ and $\beta = 1$, contradicting $\beta = 2$.

Now $z \geq 2$. If a vertex has two nonzero classes, Lemma S gives $\Pi \geq \lceil (13 + 8 + 20)/2 \rceil + h = 21 + h$. Therefore $h \geq 4$ gives $R \geq 21 + h + 1 + 2 \geq 28$. If neither vertex has two nonzero classes, the only possible $h = 3$ placement is $D + Z_u + Z_v$ in one block; without D there are at most two helpers. Thus $h \leq 3$.

For $h = 3$ and a multiple-class vertex, $\Pi \geq 24$ and $C \geq 2$. Also $A \geq 2$. For K in M this is immediate from the two untouched arrays. If K is not an edge of M, the equality $A = 1$ would require canceling its two crossing M-arrays with single Z of coefficient 2 at the two endpoints. The remaining helper lies outside those blocks and creates a nonzero outside array; it cannot attach at the inactive partner to share either crossing block. Hence $R \geq 28$.

Suppose $h = 3$ and each endpoint has only one nonzero class. The three helpers must then be D, Z_u, Z_v in a single block with source K. We have $\Pi \geq 21$ and $A \geq 2$, strengthened to $A \geq 3$ if the block group differs from M. Thus $C \leq 4$. The one-block lemma forces the two Z coefficients to be equal and every flow to vanish. The exact kernel move (K) deletes both Z columns while adding at most one E, strictly decreasing support. This contradicts the preceding matching bound $R \geq 27$.

Thus the entire Z branch reduces to $d = 0, z = 2$. The remaining $Z = 0$ branch is **one** D. This reduction explicitly covers same-source helper sharing; it does not assert that every extra helper creates a different prefix.

5. Strengthen the one-block Z lemma to $C \leq 5$. If both endpoint Z lie in one block B, there is no other Z, and $C \leq 5$, they are equal and every flow is zero. This hypothesis holds both in the $h = 3$ single-block $D + Z_u + Z_v$ use in Part H, item 4 and the $d = 0, z = 2$ use in Part H, item 6. For unequal Z, the projection right side is a nonzero two-coordinate vector on

U. Subtracting a possible multiple of c_B leaves a vector with support size 2 or 6, whereas one additional block vector c has support size 4. At least two other flow blocks are necessary. B needs two loops for unequal endpoint Z; the two Z-free flow blocks cost two each. Hence $C \geq 6$.

For equal Z, projection is 0. Distinct block vectors c are nonparallel (all 990 pairs checked). Nonzero flow therefore needs at least three blocks. If B flows, it needs at least two loops: its sole compatible E gives $\delta = 0$, and a sole L_3 cannot satisfy the S-rows for both nonzero Z coefficients. Together with two Z-free blocks this gives $C \geq 6$. If B does not flow, three Z-free blocks give $C \geq 6$ in addition to B's loop. This proves the stronger lemma.

6. Two Z: same block or same attachment. For two Z in one block, $\Pi \geq 20, A \geq 2$ gives $C \leq 5$. Part H, item 5 yields equal $Z = a$ and zero flow. Source balance (T) needs a separating endpoint, so $\Pi \geq 23$. If the block group differs from M, then $\beta = 2a \neq 0$ gives $A \geq 4$ and the budget fails. If the block belongs to M but $\beta \neq 2$, the bounds $A \geq 3$ and $C \geq 2$ also exceed the budget. The sole corner is $a = 1, \Pi = 23, A = 2, C = 2$. Here $\Gamma = 1/2$ at both endpoints. Their costs sum to at most 22, so the separating endpoint must have cost 13 and the other endpoint cost 9. The separator then supplies marginal $1/2$, but source balance (T) requires 1, a contradiction.

If the two Z lie in distinct blocks but at the same attachment, that vertex has two nonzero classes and the other active vertex has only zero class values. Their costs are therefore at least 13 and 9, respectively. Thus $\Pi \geq 23$. At most one Z-block is in M, so $A \geq 3$, and $C \geq 2$ gives $R \geq 28$. Hence distinct Z blocks must attach at different vertices.

7. Two Z: distinct blocks, different attachments.

7.1 Both outside M. Three M-arrays and two outside arrays give $A \geq 5$. The only budget is $\Pi = 20, A = 5, C = 2, X = 0$. Both active vertices have $f = 8$, are glue-free, and carry $Z = 1/2$. Every inactive vertex has $f = 5$, and each Z block has one L_3 column with $|\delta| = 1/4$. All other blocks have $\beta = 0$ and zero flow: $d = 0$ and the two Z-blocks use the entire $C = 2$ budget. For an M-edge incident to an active vertex, $H^0 \leq 1/2$. Each helper block contributes at most $1/2$ to β-weight minus divergence: source $1/2$, side $1/4$ plus at most $1/4$, absent 0. Equation (F) would give $w \leq 3/2 < 2$.

7.2 Both in M. If the two sources are distinct edges of M, then the attachment pair K is not an edge of M. The projected helper vector has nonzero antisymmetric entries on the two affected matching pairs. Every flow vector from a block in group M, however, is constant on each matching pair. An additional flowing block outside M, without Z, is therefore necessary. It costs two loops, giving $C \geq 4$. If $A = 2$, one Z is 2, forcing $\Pi \geq 22$; if $A = 1$, both are 2, forcing $\Pi \geq 23$. Both exceed 27. The only remaining budget is $\Pi = 20, A = 3, C = 4, X = 0$. Both Z coefficients equal $1/2$, every inactive center has $f = 5$, and there is one L_3 column in each Z block. The other two loops lie in a single outside flow block, denoted B'.

Let j be the unaffected edge of M. Projection requires $c_{B'}$ to be constant on j, so its group contains j and neither of the two affected matching edges. This incidence statement is checked for all 270 applicable block–group choices. Thus B' contributes no edge divergence at an affected source e. At that source, $H^0_e \le 1/2$, the helper terms contribute $1/2+1/4$, and the other matching block contributes divergence of magnitude $1/4$. Equation (F) would therefore give $w_e \le 3/2 < 2$, a contradiction.

7.3 One in M, one outside. Let the Z in group M have coefficient a, attachment u, and source $e = \{u, p\}$. Let the outside Z have coefficient b and attachment v. $A \ge 3$. For $A = 3$ the M array must cancel, so $a = 2$ and $\Pi \ge 22$. $R = 27$ forces $\Pi = 22, C = 2, X = 0$, with $f(u) \le 12, f(v) \le 9$, inactive $f = 5$. For $A = 4, \Pi \le 21, C \le 3$: both active $f \le 10$, inactive $f \le 7$. $A = 5$ forces $\Pi = 20, C = 2, X = 0$ and satisfies these same constraints. $A \ge 6$ exceeds $R = 27$. This includes $\Pi = 20, A = 4, C = 2, X = 1$ and the extra-loop $C = 3$ corner.

No Z-free nonzero-flow block can fit $C \le 3$, since the two Z blocks already cost two loops. Since $f(u) < 13$, the center u has no nonzero-marginal separator for e, so $B_{up} = 1 - 2a$. There are three possible positions for e relative to the outside group:

- If e is its source, $p = v$ and both endpoints have no separator. Then $H^0_e = 1 - a - b$, helper $= a + b$ and divergence 0, giving $w_e = 1$ instead of 2.
- If e is a side, both possible block vectors c are constant on e. The right side of the projection equation has a nonzero antisymmetric contribution from the Z in group M, and no source contribution from the outside Z. Contradiction.
- If e is absent, its divergence is zero and its helper contribution is a. Equation (F) requires $B_{pu} = 3$. An inactive p has $B \le 1$ by $f \le 7$. If $p = v$, then u lies outside its prefix and the collision-safe $f \le 10$ bound in Part H, item 3.1 gives $B_{vu} \le 2$. Neither can reach 3.

This closes every two-Z placement, including signed values, idle loops and the possible mixed column. Thus Z **cannot occur at** $R = 27, m = 2$.

8. Single D: zero flow and the private 22 obstruction. Now $d = 1, z = 0$ with $D = a \ne 0$ and source K. Nonzero flow costs $C \ge 6$. If its group differs from M, then $A \ge 4$ and $\Pi \ge 19$. If it belongs to M and $a \ne 1$, then $A \ge 3$ and $\Pi \ge 19$. If $a = 1$ in M, both $\Gamma = 1/2$ force $\Pi \ge 20$ and $A \ge 2$. All give $R \ge 28$. Hence flow is zero, and (T) applies.

A separating endpoint costs at least 13 and the other endpoint at least 8, so $\Pi \ge 22$. $\Pi = 22$ **is impossible.** If $a \ne 1/2$, both Γ are nonzero. The total line-cost budget is 42, forcing costs 13 and 9 at the active endpoints and 5 at every inactive center. The endpoint of cost 9 forces $a = 1$. The separating marginal is then $\Gamma = 1/2$, which cannot equal the source-balance requirement $\mathrm{NT} = \pm 1$.

If $a = 1/2$, then $\Gamma = 0$. A separating endpoint of cost 13 would have separating marginal zero. It must therefore cost at least 14, forcing cost 8 at the other endpoint and cost 5 at every inactive center. Three main N-lines or an extra together line cost at least 12 before a separator 5 and cannot fit 14. Thus there are precisely two main $N = -1/2$ on their main lines, with together cost at most 9, allowing any balanced collisions. Their total marginal is $0, 1/4$ or $1/2$ by F_4/F_5, and cannot equal $-NT = \pm 1$. Therefore $\Pi \geq 23$ for every single D.

9. Single D outside group M. $A \geq 4$ forces precisely $\Pi = 23, A = 4, C = X = 0$. A separator 13 and other 8 imply each inactive $f \leq 8$; hence every inactive B row is in $\{0, 1\}$, by Part H, item 3.3. If K is an M-edge, the two other M-edges are absent from the D group, and (F) asks $H^0 = 2$ despite $H^0 \leq 1$. Contradiction.

If K is not an M-edge, let j be the unaffected inactive edge of M. Unless j is a side of the D group, (F) again requires $H^0_j = 2$. If it is a side, then $H^0_j = 2 - a \leq 1$, forcing $a \geq 1$. The other side, denoted k, pairs the inactive matching partners of u and v; it does not belong to M. Both endpoints of k are light, so $H^0_k \geq 0$. But (F) at k gives $H^0_k = -a$, forcing $a \leq 0$. Contradiction. This short signed inactive-edge obstruction is also present in Part D; here its lightness premise has first been derived from $\Pi = 23$, including any balanced N at u, v.

10. Single D in M, with $a \neq 1$. $A \geq 3$ and $\Pi \geq 23$ exclude $C \geq 2$; $C = 1$ is impossible. For $X = 0$ all alternatings are in group M: the three nonzero M arrays already cost 3, and any outside zero-total array would add 2, while $A \leq 4$. The full 403-column dual (D21) excludes every such case.

The only $X = 1$ corner is $\Pi = 23, A = 3, C = 0$. The separator charge gives a total active cost of at least 22: the endpoint costs are at least $13 + 9$ if $\Gamma \neq 0$, and at least $14 + 8$ if $\Gamma = 0$. Thus inactive $f \leq 7$ and H^0 at either inactive M-edge is in $\{0, 1/2, 1\}$. Equation (F) again forces $a \in \{1, 3/2, 2\}$; the nonunit cases are $3/2$ or 2.

Positive $a > 1$ makes the other endpoint cost at least 11. The total 44 forces separator 13, other 11 and all inactive 5. Source balance $NT = 1$ then forces $\Gamma = 1$ and $a = 3/2$. The separating F_5 line has nonzero glues of magnitude 1. Such a glue cannot cross the other active vertex: that vertex has two main N-lines of cost at least 8, and a nonzero separating glue would add a line of cost at least 5, exceeding $f = 11$. It cannot cross an inactive $f = 5$ vertex either, whose glues have magnitude $1/2$. Contradiction. This proof allows a balanced N on the separator; it charges the separating glue and main lines jointly, not their stars.

11. Single D in M, with $a = 1$: costs $23, 24, 25$.

$\Pi = 23$. Both endpoints have $\Gamma = 1/2$, and their costs sum to at most 24. At most one endpoint has a nonzero-sum separator. Its cost cannot be 13, since that would give separating marginal $1/2$ rather than $NT = 1$.

Thus it costs 14 or 15 and the other endpoint at least 9. Three main lines or an additional together line would cost at least 12 before a separator 5, exceeding 15. Thus there are exactly two main $N = -1$, no additional together line, and together cost ≤ 10. All $F_4/F_5/F_6$ main marginals are nonnegative, including collisions. Their required sum is $\Gamma - \mathrm{NT} = -1/2$. The contradiction is independent of the allocation to A, C, X.

$\Pi = 24$. The bound $A \geq 2$ leaves $C \leq 1$. Since $C = 1$ is impossible, we have $C = 0$ and $A \leq 3$. All alternatings are in group M: two nonzero M arrays already need 2, and any outside zero-total array would require 2 more. $X = 1$ is allowed here.

The total line-cost budget is 46. The separating endpoint costs at least 13, and the other endpoint, whose Γ is nonzero, costs at least 9. Their combined cost is at least 22, so every inactive center has $f \leq 9$. This establishes lightness **before** using inactive-edge equations. Their B entries are 0 or 1. At each inactive M-edge, (F) gives $H^0 = 2 - a = 1$, so both directed entries are 1. Every inactive center's unique nonzero-marginal half therefore contains its inactive matching mate.

No actual star can belong to block (M, K). An active center is itself in K and cannot have source K. An inactive center's star source is its half minus the center, which contains an inactive matching mate, and therefore is not K. This conclusion uses literal sources and holds even for signed stars and the balanced-N sharing control of Part H, item 3.3. All group-M alternatings are disjoint from the four block-alternating columns of (M, K), as checked on all 45 block–group incidences. Thus both sums on the left of (B) are zero, but $\beta = 2$. Contradiction.

$\Pi = 25$ *or higher.* $\Pi = 25$ forces $A = 2, C = X = 0$ and every alternating in group M. The full dual (D21) gives $-1 = 0$. $\Pi \geq 26$ plus $A \geq 2$ already exceeds $R = 27$.

This completes the single-D branch and therefore the $m = 2$ theorem.

I. Three attachment vertices. Use the proved general identities and costs, and the helper-array rigidity $\Omega = -2e_M$ in Part D. Set $h = d + z$. Then

$$R = \Pi + A + C + X = 27, \Pi \geq 21, \Pi \geq 20 + h,$$
$$2\Pi = \sum f + 2h$$

The costs of the three inactive vertices sum to at least 15. Every active vertex costs at least 8, or 9 when $\Gamma \neq 0$. A vertex with zero total helper classes has $\Gamma = -1/2$. A surviving D requires both source endpoints in K. Thus K either contains one M-edge plus one endpoint of another, or it meets all three M-edges once. At most one M-edge lies wholly in K.

1. Three elementary facts used in the cover. **A value -2 class costs at least ten at its endpoint.** If it is the only nonzero helper class, the proved value -2 lemma gives $f \geq 11$, including balanced N classes. If another nonzero helper class is present, there are at least four N and at least one actual star, giving $f \geq 10$. We use this weaker uniform bound when extra helpers may be present.

Two different-source classes give $f \geq 12$, or $f \geq 13$ if $\Gamma \neq 0$. Each needs two N on distinct private lines; their line sets share only one line. Thus at least three nonempty N-lines occur, each costing at least four: two colliding N already cost 4. Equality 12 makes all of them F_4 and glue-free, and leaves no other nonempty line. Thus a nonzero Γ requires at least 13. This permits arbitrary balanced classes and collisions; adding them cannot erase the four nonzero main-class coefficients.

Matching-group flow vectors are constant on each M-edge. A block with group M has $c_B = 1_W - 1_V$, where U, V, W are the three edges of M. Its vector is constant on the two coordinates of each M-edge. If all Z and all nonzero flow blocks lie in M, the projected right side is a sum of antisymmetric endpoint differences, separately on the three M-edges. Equality with the left side therefore forces the endpoint Z values in each M-block to be equal. In particular, a single Z on an M-edge is impossible in such a configuration. D is absent from this projection.

2. No Z: all D cases. At least two M-arrays are untouched because at most one M-edge is inside K. Hence $A \geq 2$. Nonzero flow would need $C \geq 6$, giving $R \geq 21 + 2 + 6 = 29$. The flow therefore vanishes. We do not infer the absence of idle loops until a support budget does so.

One D. Let its coefficient be a and its source U. Source balance gives $\mathrm{NT}_U = w_U - 1 = +1$ or -1, including all balanced N at the third active vertex. (T) involves only the line sums at the source endpoints. Balanced N elsewhere, including at the third active vertex, do not enter it. A nonzero-sum separator costs at least 5 even with balanced N. At $f = 13$ its sum equals Γ because the two cost 4 main lines are glue-free. Some D endpoint has a nonzero separating marginal and $f \geq 13$, the other has $f \geq 8$, the third active vertex $f \geq 9$, and inactive $\sum f \geq 15$. Thus $\Pi \geq 24$. A D outside M would leave all three M-arrays and add its own nonzero array, giving $A \geq 4$ and $R \geq 28$. Therefore its block is (M, U).

At $\Pi = 24$, the D endpoint costs sum to at most 22. If $a \neq 1/2$, both endpoint Γ values are nonzero, so the tight endpoint costs are $(13, 9)$, with the third active $f = 9$ and every inactive $f = 5$. The proved $f = 9$ marginal law forces $a = 1$. The $f = 13$ endpoint's separating marginal is then $\Gamma = 1/2$, and the other endpoint has no separator. This contradicts $\mathrm{NT}_U = 1$.

If $a = 1/2$, an $f = 13$ endpoint cannot have a nonzero separator because its together F_4 lines are glue-free and $\Gamma = 0$. Thus the tight endpoint costs are $(14, 8)$. The $f = 14$ endpoint alone must have separating marginal 1 and together marginal -1. Its two main $N = -1/2$ together lines cost at most

nine in total, after allowing at least five for the separator. There cannot be a third main N line or another nonempty together line. The full collision-safe F_4/F_5 lemma gives total together marginal in $\{0, 1/4, 1/2\}$, a contradiction. This endpoint argument permits balanced classes at the third active vertex.

At $\Pi = 25$, the budget forces $A = 2, C = X = 0$ and $a = 1$. The endpoint cost sum is at most 24. If a separating endpoint has $f = 13$, it supplies only $1/2$, so the other endpoint would also need a separator, with combined cost at least 26. Otherwise the unique separating endpoint has $f = 14$ or $f = 15$ and supplies separating marginal 1. Its together marginal is $-1/2$. The two main $N = -1$ lines cost at most ten; extra together lines cannot fit. The collision-safe $F_4/F_5/F_6$ possibilities give only nonnegative total together marginal, contradiction. This is the main-line argument used in the earlier R2b case; it remains valid when balanced N contributions collide with the main coefficients. Larger Π cannot fit $A \geq 2$.

Two D with the same source. Each endpoint has two nonzero classes and at least four N, hence $f \geq 10$. Source balance still forces a nonzero separator at one endpoint. Its classes use at least two together N-lines costing at least eight, and the separator costs at least five, so that endpoint has $f \geq 13$. The third active vertex has $f \geq 9$. Thus

$$\Pi \geq \lceil (13 + 10 + 9 + 15)/2 \rceil + 2 = 26$$

The distinct D blocks cannot both be in M; their array count is at least three. This already exceeds $R = 27$.

Two D with different sources. The sources form a path on K. At the shared endpoint there are two different-source classes, so $f \geq 12$ by Part I, item 1. Each leaf costs at least eight; together with an inactive cost sum of at least 15, this gives $\Pi \geq 24$. If neither D belongs to group M, then $A \geq 5$. If one is in M without canceling its array, $A \geq 4$. Therefore a possible $R = 27$ identity needs $\Pi = 24, A = 3, C = X = 0$ and $D = 1$ in an M-block; call the other coefficient b.

The M leaf has $\Gamma = 1/2$ and $f \geq 9$. The shared vertex has $\Gamma = b + 1/2$, while the other leaf has $\Gamma = b - 1/2$. These two Γ values cannot both vanish. If the shared Γ is nonzero its cost is at least 13; otherwise the other leaf costs at least nine. In either event the three active costs sum to at least 30, through either $13 + 9 + 8$ or $12 + 9 + 9$. Consequently

$$\Pi \geq \lceil (30 + 15)/2 \rceil + 2 = 25,$$

contradicting $\Pi = 24$. This argument uses no prescribed value of b and does not assume that a shared $f = 12$ line configuration is feasible.

At least three D. The proved D-degree bound restated in Part B gives $\Pi \geq 26$. At least two M-arrays still need $A \geq 2$, so $R \geq 28$. This completes $Z = 0$.

3. Z present: preliminary restrictions and all $A \le 1$ cases. If $z = 1$, then $\Pi \ge 22$. When $d \ge 1$ this follows from $\Pi \ge 20 + d + z$. When $d = 0$, the two helper-free active vertices cost at least nine each, the Z endpoint at least eight, and inactive cost is 15, giving $\Pi \ge 22$ again. At least one M-array is unchanged: if K contains an M-edge, D can affect only that one and the Z at most one further array; if K meets all three once, D affects none and Z at most one. Thus $A \ge 1$. The proved bound $C \ge 5$ for a single Z already yields $R \ge 22 + 1 + 5 = 28$. This step does not require the stronger $C \ge 6$ lemma from Part G.

Hence $z \ge 2$. Support-minimality against the proved global lower 30 and the literal S rows give $C \ge 2$. Also $\Pi \ge 20 + h \ge 22$ and $\Pi \le 25$. We first rule out $A = 0$ and $A = 1$, to avoid missing high-helper boundary cases.

$A = 0$. K must meet every M-edge once, since an untouched edge has a nonzero array. All three arrays must cancel. No M-edge is contained in K, so each cancellation requires its single endpoint $Z = 2$. There are at least three helpers and each active vertex has a value -2 class. Part I, item 1 gives $f \ge 10$ at all three vertices, hence

$$\Pi \ge \lceil (15 + 30)/2 \rceil + 3 = 26,$$

already impossible with $C \ge 2$.

Thus $A \ge 1$ and $\Pi \le 24$. If $A = 1$, at least two M-arrays cancel.

$A = 1$ and K meets all three M-edges once. At least two distinct vertices carry $Z = 2$. If $h = 2$, they each cost at least eleven and the third active vertex costs nine, giving $\Pi \ge 25$. If $h \ge 3$, use costs $10, 10, 8$ to obtain $\Pi \ge \lceil (15 + 28)/2 \rceil + 3 = 25$. Therefore no $\Pi \le 24$ case fits.

$A = 1$ and $K = U \cup \{w\}$, where $U \in M$. Let V be the matching edge containing w, and let W be the fully inactive matching edge. The W-array consumes the sole alternating column. Consequently both U and V must cancel, and $Z_w = 2$ in (M,V).

If $h = 2$, the bound $z \ge 2$ makes the other helper a Z of coefficient 2 at an endpoint of U. The costs $11, 11, 9$ again give $\Pi \ge 25$. If $h \ge 4$, the value -2 endpoint costs at least ten and the other active vertices at least eight, giving $\Pi \ge \lceil (15 + 26)/2 \rceil + 4 = 25$. It remains $h = 3$.

All helpers then lie in the two canceled M-blocks. A nonzero outside β would require an additional alternating; an outside block with $\beta = 0$ would need at least two helpers besides the two cancellations. The V-block contains only its $Z_w = 2$. The U-block contains the other two helpers, either D plus Z or the endpoint Z pair. The cost at w is at least eleven and the other two active costs at least eight, so $\Pi \ge 24$. The only budget is $\Pi = 24, A = 1, C = 2, X = 0$. There are Z in both M-blocks, so both loops are confined to them. Their flow vectors are constant on M-edges, but the single Z at w gives a nonzero antisymmetric projection on V. Part I, item 1 gives a contradiction. All $A = 1$ cases are now excluded.

4. Z present: the remaining $\Pi = 22$ *and* $\Pi = 23$ *budgets.* We have $A \geq 2, C \geq 2$ and $\Pi \geq 22$. Thus only

$$\Pi = 22, \quad A + C + X = 5, \qquad \text{or} \qquad \Pi = 23, \quad A = 2,\ C = 2,\ X = 0$$

can remain. The following cases exhaust $h = d + z$ and helper placements.

$\Pi = 22$. The bound $\Pi \geq 20 + h$ forces $h = 2$, so $d = 0, z = 2$. Both Z attach at different vertices: attachment at one would cost at least ten there and nine at both other active vertices, exceeding the active budget 25. The exact costs are $f = 8$ at the two Z endpoints, $f = 9$ at the third active vertex, and $f = 5$ at every inactive vertex. Thus both Z equal $1/2$.

No M-array cancels, so $A \geq 3$, forcing $A = 3, C = 2, X = 0$. A Z block outside group M would add a nonzero array, so all Z blocks are in M. If they are distinct, the two loops are in those blocks and the projection contradiction of Part I, item 1 applies. If both Z are in one block, the proved one-block lemma gives zero flow. Source balance then contradicts the glue-free $f = 8$ endpoints: $\mathrm{NT}_U = 0$ whereas $w_U - 1 = 1$. This closes $\Pi = 22$.

$\Pi = 23, h = 2$. Again $d = 0, z = 2$. At least one M-array cancels. If the Z are in different blocks, a Z block outside group M would add an array besides the two remaining M-arrays, exceeding $A = 2$. Thus both are in distinct M-blocks, excluded by the same projected-flow argument with $C = 2$.

If the Z are in one block, the one-block lemma with $C = 2$ forces equal nonzero Z and zero flow. Its β is nonzero, so $A = 2$ then forces that block to lie in M and β to equal two. Both Z are therefore one. Source balance requires a separator, giving active costs at least $13, 9, 9$ and inactive cost 15. Including the two Z gives $\Pi \geq 25$, not 23.

$\Pi = 23, h = 3, d = 0, z = 3$. At most two Z blocks are possible because $C = 2$. If all Z are in M, one block has a lone endpoint Z, and projection gives a contradiction.

If an outside block has β zero, it contains two opposite endpoint Z. Such a block requires at least two loops, besides the loop for the remaining Z; $C \geq 3$, impossible. If an outside block has β nonzero, $C = 2$ permits at most two Z blocks. There is at most one other Z block, which can cancel at most one M-array. Two M-arrays and the outside array therefore require $A \geq 3$, contradiction.

$\Pi = 23, h = 3, d = 1, z = 2$. If both Z lie in one block, the one-block lemma makes them equal and the flow zero. If that block and the D are distinct, their arrays need at least three alternatings: at most one M-array can cancel and a nonzero outside array remains. Thus $A = 2$ forces all three helpers into the same M-block. The equal-Z kernel move deletes both Z

and creates at most one E, while modifying an existing D, strictly lowering support. That contradicts the proved global width-30 minimum.

Now put the two Z in distinct blocks. If both are in M, their two loops are confined to M and projection excludes them. If both are outside M, at most one M-array can cancel via D; the two outside Z arrays cannot both have β zero using a single D. This exceeds $A = 2$. Hence exactly one Z is in M and the other outside.

If D lies in the same M-block as the Z belonging to group M, the other two M-arrays plus the outside Z array need $A \geq 3$. If D lies in a different M-block, the only way to fit $A = 2$ is $D = 1$ and a Z of coefficient 2 in group M, canceling two M-arrays. The outside Z gives an additional class at some vertex. The value -2 endpoint costs at least ten and the other two active vertices at least eight, giving $\Pi \geq \lceil (15 + 26)/2 \rceil + 3 = 24$, contradiction.

If D is outside M, the Z in group M must have coefficient 2 to leave exactly two M-arrays. The outside β must vanish, so D and the outside Z share a block with $Z = -2D$. Its two endpoint classes are nonzero and opposite. If its source contains the M-Z endpoint, active costs are at least $10, 8, 9$; otherwise they are at least $11, 8, 8$. Either way $\Pi \geq 24$. Every $d = 1, z = 2$ placement is excluded. Finally $h \geq 4$ would force $\Pi \geq 24$, so it cannot occur in $\Pi = 23$. This completes all Z cases.

Global conclusion. Part C excludes both nonmatching width-30 budgets. For matching $R = 27$, Part F excludes zero helpers for every attachment count; S13.5, parts G, H, I and D exclude nonzero helpers at respectively one, two, three and four through six attachments. Part A lists all correction budgets, so no width-30 identity remains. The preceding lower bound $W \geq 30$ now gives $W^*(6, 2) \geq 31$.

S14. Detailed matching cases at retained support twenty-eight

The following cases complete the nonzero-helper argument of S6. All have matching corrections and $R \leq 28$. The preceding lower bound forces $R = 28$ in the global contradiction. The shared-line partner argument, arbitrary star placement, and the outside-helper allocation costs are incorporated in the relevant proofs below. For the nonmatching width-31 cases, S7's independent budget lemmas apply at $R \leq 32 - e$, including the smaller budgets here; they assume no lower bound of 32. The seven-edge argument is given in the main paper.

S14.1. **Five or six active centers.**

Scope. Matching $R = 28$, nonzero helpers, $m \in \{5, 6\}$.

Obstruction. Six centers exhaust the private budget. At five, source-degree and actual-star costs exclude $Z = 0$; array cancellations exclude $Z > 0$.

Conclusion. No such five- or six-center identity.

Assume matching corrections, $R = 28$, nonzero normalized helpers, and m in $\{5, 6\}$.

1. Exact stated inputs. Use the notation of the preceding lower-bound 31 component and (S4.8)–(S4.9), with $L_3 = L$, $h = d + z$ and $m = |K|$. The private charges give

$$\Pi \geq 15 + 2m, \qquad \Pi \geq 15 + \lceil 3m/2 \rceil + h.$$

Matching rigidity O29 applies since $R \leq 29 : \Omega = -2e_M, w = 2M$. The array totals are $\beta-2$ in the three M-blocks and β elsewhere, with $\beta = 2D+Z_u+Z_v$. Every nonzero array total needs its own alternating column; a zero-total array with nonzero support needs at least two.

Actual star mass is one at every center. Inactive $f \geq 5$. Active $f \geq 8$, or $f \geq 9$ when $\Gamma = \sum(\zeta) - 1/2$ is nonzero. Active $f = 8$ is glue-free and has a single nonzero class of value $1/2$. The proved $f = 9$ classification puts all glue in one of two class lines. At a single nonzero class with nonzero Γ and $f = 9$, its value is one and those two lines are together lines for its source. Arbitrary extra balanced classes were allowed in the proof of that classification.

S13.5, part H, item 3.2 supplies the unconditional sharing lemma: **two distinct nonzero helper classes at a center imply** $f \geq 13$, including classes with the same source and arbitrary balanced collisions. Each nonzero class requires at least two N. The four columns of one class lie on four distinct private lines; there are two columns on each h side. At a zero class, each h-side support is zero or two, so its support is even. At $Z = 0$, a vertex of D-degree $r \geq 3$ has $f \geq 4r + 2$ from the $2rN$ and at least one actual star alone.

At $Z = 0$, flow is either zero or costs $C \geq 6$; $C = 1$ is impossible directly from S. If any Z is present, normalized $C \geq 2 : aC = 1$ configuration would admit the proved support-decreasing kernel move, contradicting the already proved matching $R \geq 28$. A single Z globally costs $C \geq 6$ by S13.5, part G, Lemma L1 +. Each Z block needs a loop.

The general mixed-free equations (F), (B) and source balance (T) are used with their stated scope. In particular, when flow is zero and all nonzero classes have one source U, (T) says $\mathrm{NT}_U = w_U - 1$; balanced classes at other centers remain allowed. The block-alternating supports of (M,U), U in M, are disjoint from all twelve group-M alternating columns (the proved literal incidence check). Block-star supports form a partition; this prevents double counting in item 5.

2. A zero-helper $f = 9$ center has nonnegative actual stars. **Lemma** B_9**.** Suppose every ζ class at an active center is zero and its total cost is $f = 9$. Then it has exactly two nonzero N and every actual star coefficient at that center is nonnegative. More precisely, its nonzero actual stars are either two coefficients $1/2$ or one coefficient one. All its nonzero glues have magnitude $1/2$.

Here $\Gamma = -1/2$ requires a glue, and actual star mass one requires a star. Thus $f \geq 1 + 2 + 2N$. Balanced parity and activity imply $N = 2$. They

belong to one balanced class on one h side, with opposite coefficients on two distinct lines. Each line costs at least four; no third nonempty line fits. The costs are therefore F_4 and F_5. The F_4 line has no glue and one star equal to the negative of its N coefficient.

Let n be the N coefficient on the F_5 line, whose marginal is $r = -1/2$, and let ν be its single-position N vector. The other N is -n, so the F_4 star is n. The F_5 star vector is $l - r \cdot 1 - \nu$. Its cost is $g + 2s + 2 = 5$, giving exactly two possibilities.

- If $g = 1, s = 1$, the glue r is at one position. The N must be at a different position and cancel one of the two entries -r. Hence $n = -r = 1/2$, and the surviving F_5 star is $-r = 1/2$. Together with the F_4 star this gives two positive halves.
- If $g = 3, s = 0$, summing $l = r \cdot 1 + \nu$ gives $r = 3r + n$, so $n = -2r = 1$. The F_4 star is one. The three glues are $1/2, -1/2, -1/2$.

This proves the claim. No general positivity premise is imposed on other active vertices, on larger line costs, or on effective stars.

3. The cost of intersecting D sources when $Z = 0$. For $Z = 0$, let r_x be the number of incident nonzero D columns and assign $\pi(x) = f(x)/2 + r_x/2$. This counts each D exactly once in $\sum_x \pi(x) = \Pi$. At an active center:

$$\begin{aligned} r_x = 0 &: \pi(x) \geq 9/2, \\ r_x = 1 &: \pi(x) \geq 9/2, \\ r_x = 2 &: \pi(x) \geq 13/2 + 1 = 15/2, \\ r_x \geq 3 &: \pi(x) \geq 1 + 5r_x/2 \geq 17/2 \end{aligned}$$

The second line uses $f \geq 8$; equality has its unique D coefficient $1/2$. The third line uses the proved two-class $f = 13$ lemma. An inactive center has $\pi \geq 5/2$. Therefore

$$\Pi \geq 15 + 2m,$$

$$\text{any center incident totwo ormore } D\text{forces } \Pi \geq 18 + 2m. \tag{DC}$$

The latter statement is only asserted here for $Z = 0$. It counts the extra three units at that center once, and places every other active center at its valid $9/2$ baseline. Parallel D sources in different blocks are included, because they are distinct nonzero classes.

In particular, at $m = 5$ and $\Pi \leq 27$, all D source pairs are disjoint, so $d \leq 2$. If t of their coefficients differ from $1/2$, each such D makes both endpoint Γ values nonzero, giving the sharper bound

$$\Pi \geq 25 + t. \tag{D5}$$

Equality specifies $f = 8$ at the half-valued D endpoints, $f = 9$ at the other D endpoints and at every helper-free active center, and $f = 5$ at the sole inactive center. No assumption about the existence of these equality patterns is made.

4. Six attachment vertices. Here $\Pi \geq 27$. Z is impossible because $C \geq 2$ would give $R \geq 29$. Thus $Z = 0$.

If $\Pi = 27, (\text{ DC})$ excludes repeated incidence, and every D has coefficient $1/2$ at its $f = 8$ endpoints. Such a D cannot cancel an M-array: its β is one. All three M-arrays remain nonzero, so $A \geq 3$ and $R \geq 30$. If there were no D, the same $A \geq 3$ conclusion holds.

If $\Pi = 28$, the budget forces $A = C = X = 0$. All three M-arrays must then cancel. At $Z = 0$ each needs its $D = 1$ in that M-block. Hence $d \geq 3$ and all six endpoints have a value-one class. If there is any repeated incidence, (DC) gives $\Pi \geq 30$ already. Otherwise the three D are the three matching edges, each endpoint has $\Gamma = 1/2$ and $f \geq 9$, and $\Pi \geq 6 \cdot 9/2 + 3 = 30$. Contradiction. Larger Π cannot fit $R = 28$.

Thus every $m = 6$ case is excluded, including its zero-helper case.

5. Five attachment vertices, $Z = 0$ and a nonzero D. One matching edge contains the inactive center. No D can have that source, so its array remains -2 and $A \geq 1$. Hence $\Pi \leq 27$. Equation (DC) excludes overlapping or repeated D sources. There are one or two D on disjoint source pairs. Since $\Pi \geq 25$ and $A \geq 1, C \leq 2$. Z-free nonzero flow requires $C \geq 6$, so the flow is zero. The following support budgets also exclude idle loops whenever necessary.

5.1 $\Pi = 25$. All D coefficients are $1/2$ by (D5). Consequently $A \geq 3$, so the exact $R = 28$ budget is $A = 3, C = X = 0$. Any outside-M D would require another array, so all D lie in M. At the source U of any D, both endpoints have $f = 8$ and are glue-free, hence $H^0_U = 0$. In (F), its own β is one; the only possible other D is another M edge and contributes $1/2$ on U. Thus

$$H^0_U = 2 - 1 - (d - 1)/2 \geq 1/2,$$

contradiction. This proof uses the actual helper-modified (F) equation; it does not assert $H^0 = 2M$.

5.2 $\Pi = 26$. Now $A \leq 2$. The unchanged M-array gives $A \geq 1$, so $C \leq 1$; the S rows exclude $C = 1$, and $C = 0$ follows. At least one M-array must cancel, hence one D is the coefficient-one column in (M,U). If there is a second D, it must also lie in M: otherwise two unchanged M-arrays plus its outside array require $A \geq 3$. Bound (D5) permits at most one D whose coefficient differs from $1/2$.

With just one $D = 1$, equality forces both its endpoints to have $f = 9$. All their glues are on together class lines, so $\text{NT}_U = 0$. Source balance (T)

requires $\mathrm{NT}_U = 1$. The other three active centers can have balanced N; they do not occur in this source balance.

With two D, their coefficients are 1 and $1/2$ on two disjoint M-edges. Equality gives $f = 9$ at the first pair and $f = 8$ at the second. At either endpoint of the $D = 1$ edge U, all marginal $\Gamma = 1/2$ is on together lines, so B toward its mate is -1. Hence $H^0_U = -1$. Equation (F), with own β two and the other β one, instead requires

$$H^0_U = 2 - 2 - 1/2 = -1/2$$

Contradiction. Mixed X would not alter these equations.

5.3 $\Pi = 27$*: the star-mass obstruction.* Only $A = 1, C = X = 0$ fits. Canceling two M-arrays forces exactly two $D = 1$ on two disjoint matching edges U,V. There are no other D, by disjointness and the inactive endpoint. The four D endpoints have $f = 9$, the fifth active center has $f = 9$ with all helpers zero, and the inactive center has $f = 5$: these are forced by equality in $\Pi \geq 25 + t$ with $t = 2$.

The only nonzero alternating array is the third M-array. Since $A = 1$, its unique column belongs to group M. The proved block / group incidence therefore puts no block-alternating column in either D block (M,U) or (M,V). The literal B equation requires actual star sum two in each of those blocks, for combined sum four.

At a D endpoint, $f = 9$ confines its actual stars to its two together class lines. Their star source pairs are the containing half minus the center: they contain its D mate and exclude the center. Such a source is neither U nor V. Thus none of the four D endpoints can contribute a star to either D block. This excludes their coefficients by incidence, without assuming their stars have a sign.

The remaining active center has nonnegative actual stars by Lemma B_9, with total mass one. The inactive $f = 5$ center also has nonnegative actual stars and total mass one: its marginal is $-1/2$ and its F_5 form is a single $-1/2$ glue or three glues $1/2, -1/2, -1/2$. Their combined contribution to any subset of distinct star blocks is at most two. It cannot supply the required four. Contradiction.

The bounds $\Pi \geq 25$ and $A \geq 1$ exclude any additional Π budgets. This completes every $Z = 0$, nonzero-helper $m = 5$ case.

6. Five attachment vertices with Z present. $\Pi \geq 25$ throughout. Exactly one Z is excluded by L1 +, since $C \geq 6$ would give $R \geq 31$. Hence $z \geq 2$ and $C \geq 2$. The helper count satisfies

$$\Pi \geq 23 + h, h = d + z \geq 2$$

If $h \geq 4, \Pi + C \geq 27 + 2 > 28$. Thus $h = 2$ or $h = 3$.

For $h = 2$, there are exactly two Z and no D. They give nonzero helper values at at most two centers. At least three of the five active centers therefore have $\Gamma = -1/2$ and $f \geq 9$. The others have $f \geq 8$, and the inactive center has $f \geq 5$. Consequently

$$\Pi \geq \text{ceil } ((3 \cdot 9 + 2 \cdot 8 + 5)/2) + 2 = 26$$

With $C \geq 2$, the budget leaves $A = 0$. But two helpers can affect at most two of the three M-arrays, leaving a nonzero array. Contradiction.

For $h = 3, \Pi \geq 26$ and $C \geq 2$ again force $A = 0$. Canceling all three M-arrays uses all three helpers, precisely one per M-block, with no other helper block. Each single Z must consequently equal two; a single D equals one. Since $z \geq 2$, at least two distinct centers have a sole nonzero class of value two, and each costs at least eleven. The other three active centers cost at least eight, and the inactive one at least five. Therefore

$$\Pi \geq \text{ceil } ((2 \cdot 11 + 3 \cdot 8 + 5)/2) + 3 = 29,$$

contradiction. This exhausts Z placements, including the possibility $A = 0$; no positive lower bound on A was assumed for the general Z case.

S14.2. **Doubled-helper source degrees.**
Scope. Matching $R = 28$, $Z = 0$; repeated doubled-helper sources are retained.
Obstruction. Endpoint costs exclude two high-degree centers and the remaining large star in the source multigraph.
Conclusion. A reduction to the following one-D and two-D cases, not their exclusion.

Assume matching corrections, $R = 28$ and $Z = 0$. The doubled-helper sources are a multigraph on the active centers.

1. A tight separator with two classes on the same source. **The same-prefix cost-thirteen lemma.** Let u have two distinct nonzero helper classes with the same source pair U containing u, total line cost $f(u) = 13$, and a nonzero marginal on a line separating U. Then there is exactly one nonzero-marginal separating line, its marginal is $-1/2$, all together lines have zero glue and zero actual stars, and the two class values sum to zero. Additional balanced classes are allowed in the hypothesis but cannot occur at this cost.

The two nonzero classes require at least four N, all on together lines for U. A nonzero-marginal separating line costs at least five, including the proved collision-safe scope. Six N together with a required glue and an actual star already cost at least $12 + 1 + 2 = 15$. Five N can only come from these two classes, since another balanced class requires two additional N; they would cost at least ten on together lines, plus five on the separator. Thus there are exactly four N, all on together lines.

Their N cost eight and the separator costs at least five. Equality $f = 13$ forces the together lines to have neither glues nor actual stars, and leaves

precisely one N-free F_5 separator. Every actual star at u is on that separator, with total mass one. The N-free line formula gives star $\sum -2r$, so its marginal is $r = -1/2$. The total Γ is also $-1/2$, hence the sum of helper class values is zero. This argument uses actual star mass, without assuming star positivity.

Consequently, at a same-source two-class center with $f = 13$, its total separating marginal is either zero (if no such line exists) or $-1/2$.

2. Two attachment vertices: $d \geq 2$ is impossible. All D source pairs equal K. With $d = 2$, both endpoints have two nonzero classes, so the proved $f = 13$ sharing lemma and four inactive costs give

$$\Pi \geq (13 + 13 + 4 \cdot 5)/2 + 2 = 25$$

If K is not an M-edge, both D blocks are outside M and $A \geq 5$, already too large. If K is an M-edge, at most one D is in its M-block; the other is outside. Two other M-arrays remain unchanged and the outside array is nonzero, giving $A \geq 3$. $R = 28$ therefore forces $\Pi = 25, A = 3, C = X = 0$. Both active endpoints have $f = 13$ and all inactive vertices have $f = 5$.

The flow is zero, and all nonzero helper classes have source K, so source balance requires $\mathrm{NT}_K = w_K - 1 = 1$. The same-prefix cost-thirteen lemma restricts each endpoint's separating contribution to $\{0, -1/2\}$. Their sum cannot be one. This excludes the exact tight corner without applying the single-source 403-column dual beyond its allowed D-column scope.

For $d \geq 3$, the $2dN$ at each endpoint and one actual star give the proved charge $\Pi \geq 12 + 5d$. At most one of these distinct same-source D blocks is in M. The two other M-arrays and at least $d - 1$ outside arrays give $A \geq d + 1$. Thus $R \geq 13 + 6d \geq 31$. These estimates allow arbitrary extra loops and mixed columns; adding them only consumes support.

Thus the $Z = 0$, nonzero-helper $m = 2$ frontier has exactly one D.

3. Three attachment vertices. At $Z = 0$, assign $\pi(x) = f(x)/2 + r_x/2$, where r_x is D-degree. Inactive centers cost at least $5/2$, while active degree costs are at least

$$J(0) = J(1) = 9/2, J(2) = 15/2, J(r) = 1 + 5r/2 \text{for } r \geq 3$$

These bounds permit balanced N and are nondecreasing in r. At most one M-edge lies wholly in K, so at least two M-arrays remain untouched: $A \geq 2$.

Two D with the same source give degrees $(2, 2, 0)$. The two $f = 13$ endpoints, one helper-free active $f = 9$ and three inactive $f = 5$ yield $\Pi \geq 27$. At most one of the D blocks is in M, so $A \geq 3$ and $R \geq 30$.

For three D on three vertices, allowing parallel sources in distinct blocks, the only unordered degree patterns and their charges are

Degrees	Π lower bound including three inactive centers
$(2,2,2)$	30
$(3,2,1)$	28
$(3,3,0)$	29

For more D, select any three columns. The actual degrees are at least the selected degrees and J is nondecreasing, so $\Pi \geq 28$ persists. With $A \geq 2$ this excludes every $d \geq 3$ case at $R = 28$.

The remaining nonzero D support at $m = 3$ is one D, or two D with different sources forming a path. No closure of those cases is asserted here.

4. Four attachment vertices. The active / inactive baseline is $\Pi \geq 23$. Each center with D-degree at least two adds at least three by the same charge. Two such centers give $\Pi \geq 29$ and are already excluded at $R = 28$.

With two D sharing exactly one endpoint, the degree charge gives $\Pi \geq 26$. At most one source is an M-edge, and at least one helper block is outside M. At least two M-arrays remain unchanged and the outside array is nonzero, so $A \geq 3$ and $R \geq 29$. Two D with identical source have two degree-two centers, already excluded.

For $d \geq 3$, the only graph on four vertices with at most one vertex of degree at least two and at least three edges is the three-edge star: its degrees are $(3,1,1,1)$. Indeed all other vertices have degree at most one, so there can be at most three edges incident to the exceptional vertex, and no fourth edge can be added. This includes multigraphs: parallel edges would make both their endpoints have degree at least two.

The star has $\Pi \geq 5 + J(3) + 3J(1) = 27$. At most one of its three sources can be an M-edge. Therefore at least two D blocks are outside M and at least two M-arrays remain nonzero, giving $A \geq 4$ and $R \geq 31$. All other $d \geq 3$ configurations have at least two high-degree centers and $\Pi \geq 29$.

Thus the $Z = 0$, nonzero-helper $m = 4$ frontier has one D, or two D with disjoint sources. The proof did not assume their coefficients' signs.

S14.3. **One doubled helper with nonzero flow.**
Scope. Matching $R = 28$, $m = 2$, $Z = 0$, one D, nonzero flow.
Obstruction. The six-loop cost leaves $D = 1/2$ or 1; their divergences or opposite glue-matrix marginal sums disagree.
Conclusion. Both coefficient corners are excluded.

Assume matching corrections, $R = 28, m = 2, Z = 0$ and exactly one doubled helper, with nonzero coordinate flow.

1. Complete nonzero-flow budget cover. Suppose flow is nonzero. If the D is outside group M, its array is nonzero in addition to the three untouched

M-arrays; $A \geq 4$ then gives $R \geq 19 + 4 + 6 = 29$. Therefore D is in the block (M,U), with U an M-edge.

If $a \neq 1$, none of the three M-arrays cancels, so $A \geq 3$ and $\Pi \leq 19$. Equality forces $\Pi = 19, A = 3, C = 6, X = 0$. The private line budget is $\sum f = 2(\Pi - 1) = 36$. Four inactive centers cost at least five and the two active centers at least eight. Therefore their costs are exactly $f = 8, f = 8$, with four inactive $f = 5$, and the proved $f = 8$ law forces $a = 1/2$.

If $a = 1$, both active Γ values equal $1/2$, so $f \geq 9$ at each and $\Pi \geq 20$. Since $A \geq 2$, equality forces $\Pi = 20, A = 2, C = 6, X = 0$, with both active costs $f = 9$ and all four inactive costs $f = 5$.

These are the only two nonzero-flow corners. No coefficient grid or sign assumption is used to reduce the originally arbitrary nonzero a.

2. The $a = 1/2$ corner: three flow blocks cannot meet the demands. The two active centers are glue-free, so their B rows are zero. Every inactive $f = 5$ center has one nonzero-marginal line of marginal $-1/2$; its B row has two entries one and otherwise zero. Hence $H^0 \geq 0, H^0_U = 0$, every edge touching U has $H^0 \leq 1/2$, and every other edge has $H^0 \leq 1$. These conclusions follow from this equality budget, not from a general positivity premise on private coefficients.

Write $M = \{U, V, W\}$. The sole β value is one in block (M,U). In F the pair-node demand is

$$\text{demand}_{\,e} = H^0_e + 1[e = U] + (1[e = V] + 1[e = W])/2 - 2 \cdot 1[e \in M]$$

All three M nodes have strictly negative demand: U has demand -1, and V,W have demand at most $-1/2$. Off M the demand is $H^0 \geq 0$.

$C = 6$ allows exactly three nonzero-flow blocks, and their divergence has at most six nonzero pair-nodes. At $Z = 0$, projection says $\sum_B \delta_B \cdot (1_W - 1_V) = 0$. Summing F over edges incident to a coordinate x therefore gives

$$\sum_{e \text{containing } x} H^0_e = 1 \text{if} x \text{belongs to} U,$$

$$3/2 \text{otherwise}$$

The offmatching incident $H^0 \sum$ is consequently one at each U endpoint, and at least $1/2$ at each other coordinate. The positive offmatching edges cover all six vertices and hence number at least three. Together with the three negative M demands this saturates the maximum of six demanded pair-nodes. There are exactly three positive offmatching edges; they form a perfect matching M' disjoint from M.

At an endpoint u of U, its unique M' edge would need H^0 value one to provide the entire offmatching sum. Its other endpoint is inactive, because the only other active center is u's M mate. But every such edge has $H^0 \leq 1/2$. Contradiction.

3. The $a = 1$ corner: the private split marginals disagree. Both active centers u,v have a single nonzero helper class of value one and cost $f = 9$. There are exactly two main $N = -1$ on together lines for U, one F_4 and one F_5. All glue is on the F_5 line with marginal $1/2$. Its glue vector must be $(-1/2, 1/2, 1/2)$ up to position permutation. Indeed a one-glue F_5 line with $N = -1$ would have marginal one, whereas the no-star three-glue form has marginal $1/2$. Thus there are two positive glues and one negative glue. This is also the proved $f = 9$ equality law.

Fix u's F_5 line. Its half is $\{u, v, w\}$, where w is inactive, since every main line is together for U. All three opposite centers are inactive. The three nonzero u-glues activate their unique $f = 5$ lines in this split, each of marginal $-1/2$.

Choose a positive u-glue. At its opposite center the N-free $f = 5$ line has marginal $-1/2$ and contains a glue $+1/2$. It must therefore be the three-glue form $(1/2, -1/2, -1/2)$, not a single $-1/2$ glue. Its other two nonzero glues activate the v and w lines in the same split. The v line is its sole glue-bearing line and so has marginal $+1/2$. The w line is its unique $f = 5$ line and has marginal $-1/2$.

The sum of glue marginals on the $\{u, v, w\}$ half is consequently

$$1/2 + 1/2 - 1/2 = 1/2,$$

whereas the opposite half has total $-3/2$. Both totals sum the same three-by-three glue matrix and must agree. Contradiction. This corner is excluded by the private rows alone, without a zero-flow assumption.

S14.4. **One doubled helper with zero flow.**
Scope. Matching $R = 28$, $m = 2$, $Z = 0$, one D, zero flow.
Obstruction. $\Pi \geq 23$. Separate outside, nonunit matching and unit matching helpers; the last needs five alternating blocks with budget four.
Conclusion. All three helper locations and values are excluded.

Assume matching corrections, $R = 28, m = 2, Z = 0$ and exactly one doubled helper, with zero coordinate flow.

1. stated inputs and complete budget cover. Use the lower bound 31 proof, S13.5, parts B, D, G and H, with its exact rational row certificates. Matching rigidity gives $\Omega = -2e_M$. Write $R = \Pi + A + C + X, h = 1$, and $\sum f = 2\Pi - 2$. Source balance applies because flow is zero and all nonzero helper classes have source U:

$$\mathrm{NT}_U = w_U - 1 = 1 \text{if} U \text{is} \in M, \text{and } -1 \text{otherwise .} (T)$$

The proved private 22 obstruction is local and independent of $R = 27$: it gives $\Pi \geq 23$. Its proof in S13.5, part H, item 8 uses only (T), endpoint costs and the collision-safe $F_4/F_5/F_6$ laws. The same proof gives an endpoint cost sum at least 22 : a nonzero $-\Gamma$ separator costs at least 13 and the other

endpoint at least 9; with $\Gamma = 0$ it costs at least 14 and the other endpoint at least 8.

An inactive center with $f \le 9$ has one nonzero-marginal N-free line of marginal $-1/2$, all its B entries are zero / one, and every actual star has source equal to that half minus its center. A nonempty zero-marginal line with a glue costs at least 6. The F_5 nonzero-marginal law says all nonzero glues have magnitude equal to the marginal's magnitude, including the proved scope with N collisions.

The complete coarse support cover is:

D location / value	Possible Π	Additional budget facts
Outside group M	$23, 24$	$A \ge 4$; $C = 0$ since $C = 1$ is impossible and $C \le 1$.
In $M, a \ne 1$	$23, 24, 25$	$A \ge 3$; $\Pi = 25$ forces $A = 3, C = X = 0$.
In $M, a = 1$	$23, 24, 25, 26$	$A \ge 2$; $\Pi = 26$ forces $A = 2, C = X = 0$.

$C = 1$ is excluded by the Z-free S rows. All idle-loop budgets with $C \ge 2$ are retained until excluded below. Any larger Π already exceeds 28 with its required arrays. These rows include all mixed support slack.

The 403-column dual D21 is used only when $C = X = 0$ and every alternating belongs to group M; it then gives $-D + t_U^M/2 = -1 = 0$. Its allowed columns are all private columns, all N attached at U, all twelve group-M alternatings and this one D. No extension to mixed columns, loops or outside-M alternatings is asserted.

2. A useful unit-class endpoint bound. **Lemma U25.** If $a = 1$ in M and flow is zero, then

$$f(u) + f(v) \ge 25$$

Suppose the sum is at most 24. At least one endpoint has a nonzero separating marginal. Both endpoints have $\Gamma = 1/2$ and $f \ge 9$, so at most one can be separating, since each separator costs at least 13. Its separating sum is therefore $\mathrm{NT} = 1$. Cost 13 would make its two together main lines F_4, with total marginal zero, and its separator $\sum \Gamma = 1/2$. Hence its cost is 14 or 15.

At such a center three main N-lines, or an additional nonempty together line, would cost at least 12 before the separator's cost at least 5. Thus there are exactly two main $N = -1$, on two together lines whose combined cost is at most 10. Their possible $F_4/F_5/F_6$ marginals are all nonnegative, including balanced collisions. But their required sum is $\Gamma - \mathrm{NT} = -1/2$.

Contradiction. This is the proved S13.5, part H, item 11 ($\Pi = 23$) argument with its purely local endpoint hypothesis stated explicitly.

3. D outside group M. $\Pi \le 24$ and the endpoint sum at least 22 imply every inactive $f \le 9$:

$$f(x) \le 2 \cdot 24 - 2 - 22 - 3 \cdot 5 = 9$$

Thus all inactive B entries are zero or one. If U is an M-edge, the two other M-edges are absent from the D group: a different perfect matching containing U cannot contain one of them without being M. Their F equations require $H^0 = 2$, whereas $H^0 \le 1$.

If U is not an M-edge, let e be the inactive M-edge. Unless it is a side of the D group, F again requires $H^0_e = 2$. If it is a side, then $H^0_e = 2 - a \le 1$ gives $a \ge 1$. The other side j joins the two inactive M-mates of u,v and is not an M-edge. Its endpoints are light, so $H^0_j \ge 0$, whereas F requires $H^0_j = -a$. Thus $a \le 0$, a contradiction.

This excludes $\Pi = 23$ and 24, including X and every permitted alternating count. It does not assume the signs of the inactive glues or of a.

4. In M, nonunit coefficient. At $\Pi = 25$ the budget is $A = 3, C = X = 0$. All three nonzero M-arrays require a column, so every alternating is in M. D21 applies and gives a contradiction. It remains $\Pi = 23$ or 24; no restriction on their permitted C or X is needed for the following private argument.

The endpoint sum at least 22 gives inactive $f \le 9$. At either inactive M-edge F requires $H^0 = 2 - a$, while lightness gives H^0 in $\{0, 1/2, 1\}$. Since $a \ne 1$, necessarily $a = 3/2$ or 2. A center with this sole nonzero class has $f \ge 11$: the proved $f \le 10$ class-value list has no positive values other than $1/2$ and 1. A separating center has $f \ge 13$.

4.1 $\Pi = 23$. Total cost 44 forces endpoint costs $13, 11$ and all inactive $f = 5$. The separating endpoint alone supplies NT $= 1$; at $f = 13$ its separating sum is $\Gamma = a - 1/2$, forcing $a = 3/2$. Its separating F_5 line has glues of magnitude one. Such a glue cannot cross the other active center: there a nonzero separating line plus two main lines costs at least 13, and a nonempty zero-marginal separating line costs at least 14, exceeding $f = 11$. It cannot cross an inactive $f = 5$ line either, whose glues have magnitude $1/2$. Contradiction.

4.2 $\Pi = 24$. Total cost 46 gives endpoint sum at least 24. Either all inactive costs are 5 and the endpoint sum is 26, or one inactive cost is 7, the others 5, and the endpoint sum is 24. There is no other inactive excess at most 2.

If both endpoints have nonzero separators, they must both have cost 13. Their separating sums are both $\Gamma = a - 1/2$, totaling 2 or 3 instead of NT $= 1$. Hence there is only one separating endpoint, of cost $13, 14$ or 15.

If its cost is $13, \mathrm{NT} = 1$ forces $a = 3/2$ and its F_5 separator has all nonzero glues of magnitude one. The other endpoint has cost at most 13 and no nonzero separator; a zero-marginal separating glue line would cost at least 6 in addition to its two main lines, so cannot fit either. Thus the separator's glues must cross inactive centers. All but at most one are $f = 5$ and cannot take such a glue. The exceptional $f = 7$ line, if present, is N-free with marginal $-1/2$ and glue form

$$(-1/2, b, -b), \text{with } b\text{nonzero and } b \neq \pm 1/2,$$

up to permutation. A glue of magnitude one therefore forces another glue of magnitude one to a distinct opposite center. Because the first line separates u,v, this opposite half contains u but not v; its other centers are inactive $f = 5$. None can supply that second glue. Contradiction.

If the separator costs 14 or 15, the other endpoint costs at most 12 and has no separating glue. No exceptional inactive $f = 7$ fits: these endpoint costs would leave an inactive excess of at most one. Thus all separator glues cross inactive $f = 5$ lines and belong to $\{0, \pm 1/2\}$. Only one nonzero-marginal separator can fit, since two cost at least 10 before the two main lines' cost 8. Its required marginal is one.

On the forced half-grid, marginal one requires exactly two glues $+1/2$. Its effective star vector has three nonzero entries $(-1/2, -1/2, -1)$. Each N can affect at most one star position, so actual-star count plus N count on that line is at least three. Its cost is therefore at least $2 + 2 \cdot 3 = 8$, allowing arbitrary N collisions. But the available separator cost is at most $15 - 8 = 7$. Contradiction.

This excludes all nonunit cases without extending D21 to their C/X or outside-alternating slack.

5. In $M, a = 1$: all but the four-alternating corner. U25 excludes $\Pi = 23$. At $\Pi = 24$ or 25 it gives inactive

$$f(x) \leq 2\Pi - 2 - 25 - 15 \leq 8$$

Hence every inactive center is light. At an inactive M-edge, F gives $H^0 = 1$, forcing both directed B entries one. Each inactive center's unique nonzero-marginal half contains its M-mate. It has no additional nonempty line, since $f \leq 8$ and a nonempty zero-marginal line costs 6.

No actual star belongs to any block with source U: active centers are in U and cannot have source U, and an inactive center's star source contains its inactive M-mate and so is not U. In particular the actual star sum of $B_U = (M, U)$ is zero. Its B equation therefore needs block-alternating sum two.

If $A \leq 3$, every alternating belongs to group M: the two nonzero M-arrays already need two columns, and an outside zero-total array would need two more. Group-M alternatings are disjoint from the alternating

support of B_U. Contradiction, irrespective of the remaining idle loops or mixed columns. This covers all $\Pi = 25$ budgets and all $\Pi = 24$ budgets except $\Pi = 24, A = 4, C = X = 0$.

$\Pi = 26$ forces $A = 2, C = X = 0$, all alternatings in M, and D21 applies. Larger Π cannot fit. Thus the unique remaining support tuple is

$$\Pi = 24, A = 4, C = X = 0, D = 1 \in (M, U).(Q)$$

6. Equality facts at Q, including balanced N. At Q, U25 and inactive $f \geq 5$ force every inactive $f = 5$: each has $f \leq 6$, and $f = 6$ is impossible. The two endpoint costs sum to 26. They are either both 13 and separating, or the unique separating endpoint has cost $14, 15, 16$ or 17 with its mate of cost $12, 11, 10$ or 9 respectively.

Costs 14/15 are excluded by the U25 main-line marginal argument. A value-one center cannot have $f = 10$ by the proved S13.5, part D, item 2 lemma, so the 16/10 pair is impossible. At 17/9, the $f = 9$ mate has all glue on a together line, and all inactive glues have magnitude 1/2. The separator at the $f = 17$ endpoint therefore has a forced half-grid and marginal one. There is exactly one such line, costing at least 8 as above. The two main lines then have combined cost at most 9 and nonnegative marginal; no extra together line fits. They cannot have required $\sum -1/2$.

Thus both endpoints have cost 13 and separating marginal 1/2. Each has exactly two main together F_4 lines and one separating F_5 line. The two main N are -1. Arbitrary balanced N reduce to exactly two possible forms at each endpoint:

- No balanced N. Both main lines have actual star $+1$. The separating line is N-free, has marginal 1/2 and nonpositive actual stars.
- One balanced pair. Its $+1$ cancels a main $N = -1$ in the same V cell, leaving that F_4 line star-free; its -1 occurs on the separating F_5 line. The other main line has star $+1$. The separator has no actual stars and three glues $(-1/2, 1/2, 1/2)$.

To justify exhaustiveness, six N plus a glue and an actual star already cost 15. Three main N would use three together lines before the separator and cost at least 17. Hence there are two or four N, with the extra two a balanced pair. They cannot cancel both main N, since they have opposite coefficients. A main F_4 line containing two N must have zero actual stars and zero glue, forcing its extra coefficient $+1$. Its partner on the F_5 separator is -1, giving the second form above. There is no room for an additional nonempty line. This retains the known balanced sharing phenomenon rather than assuming two main stars survive.

Balanced bridge incidence. If a balanced $+1$ cancels a main N in one of these together V cells, its unique partner on the same h side lies on the half consisting of the center and the other M-pair containing that line's third vertex. Its V position is opposite the center's U-mate. This is checked

for all 360 canonical N columns, by the literal V and h-class incidences. Consequently, in form 2 the negative separator glue is the glue between the two active centers, whenever that separator is forced by this canceled line.

7. The remaining alternating pair forces too many charged blocks. The two nonzero M-arrays each require a column. To give B_U alternating sum two, at least one outside-M alternating is necessary. Its zero-total array requires a second. At $A = 4$ there are therefore exactly two group-M columns, each of coefficient -2, and two columns in one outside array, with opposite coefficients. The four columns of an array have four different block owners, a literal incidence fact. Exactly one of the outside pair belongs to B_U, with coefficient $+2$; its companion is -2.

Every other alternating coefficient is -2. Thus each block other than B_U, where $\beta = 0$, must have nonnegative actual star sum. The array containing the selected B_U alternating has two possible types of other block owner: another block (M,V) for V in M minus U, or a block whose source consists of one active and one inactive vertex. This ownership statement is checked on all 45 choices of M,U and all four B_U columns.

If the companion belongs to (M,V), its B equation requires positive star sum. But all possible stars with source V come from active separating lines: inactive halfmates include their own M-mates, and main together stars have an active mate in their source. Active separator stars are nonpositive by item 6. Contradiction.

Otherwise write the companion's source as $\{u, w\}$, with u active, w inactive, v the other active center, and x the inactive M-mate of w. Only two centers can supply stars in this block: v on a main together line and x on its unique inactive line. Each contributes at most one. Indeed v has at most one main line on half $\{u, v, w\}$, and all inactive $f = 5$ star coefficients are $1/2$ or 1. The companion alternating is -2, and any additional alternating in this block is also negative. Its B equation therefore requires at least two. Both stars must equal one, with no additional alternating in that block.

The inactive x line has half $\{u, w, x\}$. Its star $+1$ is the dense N-free $f = 5$ form, whose positive glue is opposite center v. This last position is another literal ownership incidence of the same companion pair. It activates v's separator on the complementary half consisting of v and the remaining inactive M-pair V. Every separator star at v thus has source V. No other center can contribute a positive star with that source, whereas all block-star sums must be nonnegative. Consequently the separator has no actual stars: v has form 2 of item 6.

The balanced bridge incidence makes its glue to u equal $-1/2$. This activates u's separator on $\{u, w, x\}$. Its source is the inactive M-pair $\{w, x\}$; the same nonnegative-block argument forces u to have form 2 too. Both active separator lines have negative glue to each other and two positive glues to inactive centers. The four inactive $f = 5$ lines are therefore all

dense. After ordering active centers first, the entire private matrix is forced to be

$$(1/2) \cdot [[-1, 1, 1], [1, -1, -1], [1, -1, -1]]$$

Each of the four inactive centers has one actual star $+1$. Their source pairs are four different pairs, each containing one active vertex and an inactive M-mate. Thus their block owners are distinct, and none is B_U. All actual stars are now nonnegative: the active centers each retain one main star $+1$ and have no separator stars.

The four positive inactive blocks each need an alternating column by their $\beta = 0B$ equation. B_U needs a further column because its star sum is zero and $\beta = 2$. These are five distinct block supports, giving $A \geq 5$, contrary to $Q'sA = 4$. This count is solely a block-partition count; no group-array count is added to overlapping block counts.

S14.5. **Three or four centers with doubled helpers.**
Scope. Matching $R = 28$, $Z = 0$, three or four active centers, after the source-degree reductions.
Obstruction. Endpoint costs remove one D, shared-center costs remove a two-D path, and actual-star demands remove disjoint sources.
Conclusion. No remaining three- or four-center identity.

Assume matching corrections, $R = 28, Z = 0$ and m in $\{3, 4\}$, after the source-degree exclusions.

1. Three local tools.

E23: one-source endpoint cost. Suppose exactly one $D = a$ has source $U = \{u, v\}$, flow is zero, and any additional N have zero helper classes. The proved source balance says $\mathrm{NT}_U = w_U - 1$, equal to 1 or -1, allowing balanced N at any center. Then

$$f(u) + f(v) \geq 23. \text{(E23)}$$

For $a \neq 1/2$, both $\Gamma = a - 1/2$ are nonzero. A nonzero separating marginal costs at least 13 at one endpoint and the other costs at least 9. If the sum were at most 22, the costs would be $13, 9$. The $f = 9$ law forces $a = 1$, whereas the $f = 13$ separator contributes $\Gamma = 1/2$, not $\mathrm{NT} = \pm 1$.

For $a = 1/2$, an $f = 13$ separator would contribute $\Gamma = 0$. Thus a nonzero separator costs at least 14, and the other endpoint at least 8. At total 22 the costs must be $14, 8$. Three main lines or an extra together line would already cost 12 before a separator 5. There are exactly two main $N = -1/2$ together lines, with total cost at most 9. Their $\mathrm{F}_4/\mathrm{F}_5$ marginal sum is $0, 1/4$ or $1/2$, never the required $-\mathrm{NT} = \pm 1$. This proves E23.

If $a = 1$ and U is an M-edge, the previously recorded U25 bound improves this to $f(u) + f(v) \geq 25$. Its proof is likewise local and permits balanced N elsewhere; it is not confined to $m = 2$.

B_{10} : *a helper-free active center of cost at most* 10. If every ζ at an active center is zero and $f \leq 10$, all actual stars are nonnegative, every positive glue is at most $1/2$, and f is 9 or 10. At $f = 9$ the earlier B_9 classification gives B entries zero / one and all nonzero glues have magnitude $1/2$.

To prove the new $f = 10$ assertion, $\Gamma = -1/2$ requires a glue and actual star mass 1 requires a star. Balanced parity leaves exactly two N: four N plus a glue and a star cost at least 11. They form one balanced pair, with coefficients n,-n on distinct lines. Each line costs at least 4; no third nonempty line can fit. At total $10, F_4 + F_6$ has marginal 0, so the only possibility is $F_5 + F_5$.

A one $-NF_5$ is either a one-glue line of marginal -n with one actual star n, or a three-glue line of marginal $-n/2$ with no actual stars. Two of the first kind have star $\sum 0$, and two of the second kind have no stars. Thus there is one of each. The star-bearing line must have $n = 1$, so its sole glue is -1. The other line has $N = -1$, marginal $1/2$, glues $(-1/2, 1/2, 1/2)$, and no stars. The sole actual star is $+1$. The exact affine-mask check permits arbitrary rational n and glues.

P14: positive two-class centers of cost at most 14. Suppose a center has two nonzero classes with distinct source prefixes, values $(1/2, 1/2), (1, 1/2)$, or $(1, 1)$, and $f \leq 14$. Then every private marginal at that center is nonnegative, and every B entry is nonpositive.

Each class requires two N. Six N plus a required actual star cost 14 and leave no glue; this would give $\Gamma = 0$, and the glue-free actual star mass would be twice the sum of the two class values. In the applications below those values are $(1/2, 1/2), (1, 1/2)$, or $(1, 1)$, so this cannot equal 1. Equivalently their Γ is strictly positive and a glue is required, making six N cost at least 15.

Five main N would use at least four lines: distinct-prefix class line sets share only one line. Their cost is at least 16. Extra balanced classes require two N. Therefore in these applications there are exactly four main N, on exactly three lines, with all specified N coefficients negative. Every line costs at least 4 and hence at most 6. F_4 and F_6 have marginal 0; the collision-safe F_5 law with any specified negative N gives positive marginal. There is no room for another line. Thus every marginal is nonnegative. The assertion P14 is used here only for the three positive value pairs explicitly listed.

2. One D at $m = 4$. The other two active centers have zero helper classes and cost at least 9 each; the two inactive centers cost at least 5 each. E23 gives

$$\Pi \geq \text{ceil } ((23 + 18 + 10)/2) + 1 = 27$$

One D leaves at least two M-arrays, so $A \geq 2$. This gives $R \geq 29$ and excludes $m = 4, d = 1$, including mixed or idle-loop slack.

3. One D at $m = 3$. The third active center w is helper-free with $f(w) \geq 9$, and inactive cost is at least 15. E23 gives $\Pi \geq 25$. A D outside M needs $A \geq 4$ and is impossible at $R = 28$. Hence D lies in $(M, U), U = \{u, v\}$ in M.

3.1 Nonunit a. $A \geq 3$ leaves only $\Pi = 25, A = 3, C = X = 0$. The endpoint cost is at least 23, so every inactive $f \leq 48 - 23 - 9 - 10 = 6$ and hence is 5. There is an M-edge W with two inactive endpoints. Its F equation gives $H^0_W = 2 - a$, while the $f = 5$ endpoints give H^0_W in $\{0, 1/2, 1\}$. Thus a is $1, 3/2$ or 2. The nonunit values make each endpoint cost at least 11. The total 48 then forces endpoint costs $13, 11$, third active $f = 9$ and all inactive $f = 5$.

The $f = 13$ separator is the only nonzero separator; NT $= 1$ forces its $\Gamma = a - 1/2$ to equal 1 and hence $a = 3/2$. Its F_5 glues have magnitude 1. The other endpoint cannot have any separating glue at $f = 11$. All other centers, including the helper-free $f = 9$ center by B_9, have glue magnitude $1/2$. The separator has nowhere to cross. Contradiction.

3.2 Unit $a = 1$. U25 gives $\Pi \geq$ ceil $((25+9+15)/2)+1 = 26$. Since $A \geq 2$, the only budget is $\Pi = 26, A = 2, C = X = 0$. All alternatings belong to M: an outside zero array needs two columns in addition to the two nonzero M-arrays. Thus the D block has no block-alternating column and needs actual star $\sum 2$.

The total line cost is 50. Every inactive f is 5, because U25 leaves at most one excess unit. The only endpoint / third-center cost pairs are

$$f(u) + f(v) = 25, f(w) = 10; \text{or} 26, f(w) = 9$$

In the first pair, both endpoints cannot be separating, since that costs 26. A unique $f = 13$ separator supplies $1/2$ rather than NT $= 1$; a unique $f = 14, f = 15$ separator is excluded by the same nonnegative main-marginal argument used for U25. Unit-class $f = 10$ is impossible. Thus the only endpoint pair left is $16, 9$, with a single separator of marginal 1 at the first endpoint. Its glue to the $f = 9$ mate is zero, since the mate's glue lines are together lines. Each of the other two entries is at most $+1/2$: inactive $f = 5$ has that bound, and the third active center does by B_{10}. To sum to 1, the separator must have exactly two glues $+1/2$. Its effective star vector has three nonzero positions. Each N can cancel at most one, so glue cost plus twice(actual stars $+N$) is at least $2 + 2 \cdot 3 = 8$. The two main lines then have total cost at most 8 and zero marginal, contrary to their required marginal $-1/2$.

In the second pair, the helper-free center has $f = 9$, so B_9 gives its B entries zero / one and nonnegative stars of total mass 1. All inactive B rows also have entries zero / one. The two M-edges other than U have $H^0 = 1$, forcing both directed entries 1. Every inactive center's half therefore contains its M-mate, and none of its stars can have source U. Neither u nor

v can have source U. Only w can contribute to the D block, and its total nonnegative mass is 1, less than the required 2. This closes every $m = 3$ single-D case.

4. Two D sources forming a path at $m = 3$. The shared endpoint has $f \geq 13$ by the proved two-class lemma, each leaf $f \geq 8$, and inactive cost 15, so $\Pi \geq 24$. Two outside-M D need $A \geq 5$ and cannot fit. One D is therefore in (M,U), coefficient a, and the other, coefficient b, has a different intersecting source and lies outside M. Write $U = \{u, v\}$, the other source $\{u, w\}$, with u shared.

If $a \neq 1, A \geq 4$ forces $\Pi = 24, A = 4, C = X = 0$. Equality gives shared $f = 13$, both leaves $f = 8$ and every inactive $f = 5$. The leaf laws force $a = b = 1/2$. P14 makes $B_{uv} \leq 0$; the glue-free leaf v has $B_{vu} = 0$. But F on U, which cannot be a side of a block with intersecting source $\{u, w\}$, needs $H^0_U = 2 - 2a = 1$. Contradiction.

If $a = 1$, the v leaf costs at least 9 and $\Pi \geq 25$. The exact budget is $\Pi = 25, A = 3, C = X = 0$. Its line $\sum 46$ leaves only two active cost patterns:

$$(f(u), f(v), f(w); b) = (14, 9, 8; 1/2), \text{or} (13, 9, 9; 1)$$

Indeed inactive $f = 6$ and unit-class $f = 10$ are impossible, and an $f = 8$ or nonzero $-\Gamma f = 9$ leaf forces the displayed value. P14 applies in both patterns and gives $B_{uv} \leq 0$. The unit $f = 9$ leaf has $B_{vu} = -1$ because all of its marginal $1/2$ is on a together line. Hence $H^0_U < 0$, whereas F requires $H^0_U = 2 - 2 = 0$. This excludes every path case.

5. Two disjoint D sources at $m = 4$. Let sources be U,V and coefficients a,b. Each endpoint has just one nonzero class. There are two inactive centers. Thus $\Pi \geq 23$. At $\Pi \leq 25$, the line sum is at most 46, only four above the baseline active 32 plus inactive 10. No endpoint can reach separating cost 13, so all separating marginal sums $\mathrm{NT}_U, \mathrm{NT}_V$ vanish.

The source equations obtained directly from F are

$$w_U = 1 + \mathrm{NT}_U + b \cdot 1[U \mathrm{is a side of the } V - \mathrm{block}\ ],$$
$$w_V = 1 + \mathrm{NT}_V + a \cdot 1[V \mathrm{is a side of the } U - \mathrm{block}\ ]. \quad \text{(T2)}$$

Their own helper contributions cancel $1 - 2a$ or $1 - 2b$ from the endpoint marginals. Since each w is 0 or 2, both side incidences must hold and a,b belong to $\{-1, 1\}$. Two disjoint sources determine a unique matching, so both D are in the same group. Value 1 needs $f \geq 9$ at each endpoint; value -1 needs $f \geq 10$ by the proved $f \leq 9$ laws. $\Pi \leq 25$ therefore forces $a = b = 1$, their group $M, \Pi = 25$, all four active $f = 9$ and both inactive $f = 5$.

5.1 The forced $\Pi = 25$ *private core needs* $A \geq 4$. Every active $f = 9$ glue line is dense, with marginal $1/2$ and glues $(-1/2, 1/2, 1/2)$, and its half contains its own D source. Its third vertex cannot be an endpoint of the other D: the opposite active mate would then receive a glue on a separating line, impossible at $f = 9$. Thus the third vertex is inactive. The opposite half consists of the other D source plus the other inactive vertex. The dense opposite active rows then activate every row in this partition.

The main N on a together half has its V position opposite the M-mate of that half's third vertex, a literal incidence checked for all $360N$. The two active rows and two active columns consequently have negative glue at the inactive coordinate. Ordering the inactive coordinate last, the forced matrix is

$$(1/2) \cdot [[1, 1, -1], [1, 1, -1], [-1, -1, 1]]$$

The two inactive stars are $+1$, one in each D block. All active F_5 stars are zero; their remaining F_4 stars have sources containing their own mate and contribute to neither D block. Hence each D block needs block-alternating $\sum +1$.

The remaining M-array has nonzero total and needs a group-M column, which belongs to neither D block. With $A \leq 3$, outside-M columns can only be an opposite-coefficient pair in one zero-total array. Since each alternating has exactly one block owner, that pair cannot supply $+1$ to both distinct D blocks. All-group-M columns cannot supply either. Thus $A \geq 4$ and $R \geq 29$. This handles every C/X budget at $\Pi = 25$.

5.2 $\Pi = 26$. $A \leq 2$ forces at least one canceled M-array. Its D is 1 in (M,U). The second D must also be in M: an outside D would require its array and the two unchanged M-arrays, giving $A \geq 3$. Call its coefficient b. Since $A \geq 1, C \leq 1$ and the Z-free S rows exclude $C = 1$, so $C = 0$.

If $b \neq 1/2$, all four endpoints have nonzero Γ and $f \geq 9$. The line $\sum 48$ leaves only two excess units, so none can have a separator. Equation T2 then forces $b = 1$. The proved two-unit-D obstruction for $\Pi \leq 26, A \leq 2, C = 0$ excludes the case, including mixed slack.

If $b = 1/2$, T2 requires $\mathrm{NT}_U = 1/2$. The baseline costs are $9, 9, 8, 8, 5, 5$, with four units of slack. Exactly one unit endpoint must be a separator at $f = 13$, with all others at baseline. Its F_5 separator has marginal $1/2$. The other unit endpoint cannot receive a separating glue; the two half-valued endpoints are glue-free. Thus its F_5 line cannot be dense and must be a single positive glue $+1/2$ to an inactive center x. The inactive $f = 5$ line at x is then dense and forces nonzero glues to the other two vertices on the separator's half. At least one is a glue-free half-valued endpoint: after excluding the other unit endpoint and x, only one other inactive vertex is available. Contradiction.

5.3 $\Pi = 27$*: extension of the two-unit-D obstruction.* The only budget is $A = 1, C = X = 0$. Both D must be 1 in M to cancel two arrays. The only alternating belongs to the third M-array, so the D blocks need actual star mass 2 each. The line $\sum 50$ has four excess units above active $f = 9$ and inactive $f = 5$.

An active endpoint can have a star with either D source only on a separating line for its own source. Such a line forces $f \geq 13$: two main lines cost 8, any additional nonempty $N/$ star line costs at least 4, and equality 12 would be glue-free, contradicting $\Gamma = 1/2$. A separator without N instead costs at least 5. Thus at most one active endpoint can contribute, and if it does, it has $f = 13$ and all other centers are at their minimum.

For that endpoint, T2 gives total separating marginal 0. Two nonzero separators would cost at least 10 beyond the main cost 8. A nonempty zero-marginal line with a glue costs at least 6, also too much. Hence the relevant separator is glue-free, of cost 4. The two main lines are F_4 and F_5, with main marginal $1/2$. The F_5 is the dense, star-free single-N form. To have an actual separator star, a balanced $+1$ cancels the $N = -1$ on the F_4 main line, with its partner -1 on the separator. The separator star is therefore $+1$. These conclusions follow from the V cells and balanced parity; no surviving pair of main stars is assumed. That one active center contributes at most 1 to the two D blocks, while the two inactive $f = 5$ centers contribute at most 2. This cannot provide their required total 4.

Otherwise no active endpoint contributes to either D block. Every inactive $f \leq 9$ has one nonzero-marginal line of marginal $-1/2$ and no other nonempty line. Its star source is fixed, so each inactive center can contribute to at most one D block. Both must therefore contribute exactly 2, one to each block. A star of coefficient 2 on an N-free line of marginal $-1/2$ requires its corresponding glue to equal $3/2$. Both inactive centers thus have $f \geq 7$. The four excess units force both $f = 7$ and all active $f = 9$. But an $f = 7$ line with marginal $-1/2$ has glue form $(-1/2, t, -t)$; a glue $3/2$ forces another glue $-3/2$ at a distinct opposite center. Only one opposite center is inactive; the other large glue would hit an active $f = 9$ line, whose glue magnitudes are $1/2$. Contradiction.

$\Pi \geq 28$ cannot fit $A \geq 1$. This exhausts the disjoint-source case.

S14.6. **Two centers with physical helpers.**
Scope. Matching $R = 28$, two active centers, $Z > 0$.
Obstruction. Remove one Z and excess helpers; split the remaining placements by block, attachment and matching membership. Projection and source flow give the contradictions.
Conclusion. Every placement is excluded, with idle loops included.

Use the lower bound 31 notation and equations. $K = \{u, v\}, h = d + z, \Pi \geq 19, \Pi \geq 18 + h, R = \Pi + A + C + X, 2\Pi = \sum f + 2h$. At least one M-edge misses K, so $A \geq 1$. Inactive $f \geq 5$ with no $f = 6$; active $f \geq 8$, or

9 if $\Gamma \neq 0$. A vertex with two nonzero classes costs at least 13. Actual star mass is 1 at every center. Every surviving D has source K.

1. Local refinements and projection controls.

1.1 Two classes at cost 13 *have equal absolute values.* Suppose a center has two nonzero classes a,b and $f = 13$. Then $|a| = |b|$. Six N plus an actual star already cost 14. Five main N occupy at least four lines for distinct prefixes; for the same prefix the proved different-position incidence makes their minimum cost $5 + 5 + 4 = 14$. Thus exactly four N remain, two per class, and no balanced class fits.

For three occupied lines, their costs are $4, 4, 5$. The shared two-N line must have F_4 or F_5. F_4 requires opposite coefficients at the same V position, giving $a = -b$; F_5 requires equal coefficients at distinct V positions, giving $a = b$. Two occupied lines can occur only for equal prefixes, with different V positions on both lines. Each costs at least 5. At total 13, at least one costs 5 or 6; the former requires $a = b$, the latter $a = -b$. Four occupied lines cost at least 16. This proves the claim, including the same-prefix case.

1.2 Values $2, 2$ *or* $2, -2$ *cannot occur at* $f = 13$. In the three-line case, all marginal lies in an F_5 line and belongs to $\{\pm 1, \pm 2\}$. It cannot equal $\Gamma = 7/2$ for $(2, 2)$, or $-1/2$ for $(2, -2)$.

In the two-shared-line case, values $(2, 2)$ require costs $5, 8 : \mathrm{F}_6$ is incompatible with equal values and the proved shared-line F_7 law excludes equal values. The F_5 has marginal 2 and no stars; the F_8 would need marginal $3/2$ and actual star mass 1. With two $N = -2$ at distinct positions, F_8 has either zero glues and two stars, or two glues and one star. The first has marginal 0. For the second, writing $S = l - (3/2)1 + (2, 2, 0)$ shows that making any two star positions zero forces all three glue positions nonzero. Thus it is not F_8.

Values $(2, -2)$ require costs $6, 7$. The F_6 has marginal 0 and no stars. The F_7 must have marginal $-1/2$ and actual star mass 1. Its three-glue, zero-star option fails the mass. Its one-glue, one-star option also fails: inserting a single glue $-1/2$ into $S = l + (1/2)1 + (2, -2, 0)$ always leaves at least two stars. These finite scalar assertions are checked exactly for every position choice.

1.3 A single class at $f = 11$ *and the outside-prefix entry* $B = 3$. For a sole class $a = +2$ or -2 at $f = 11$, all marginal lies on one F_7 main line, and every B entry toward a vertex outside its prefix is either 0 or $1 - 2a$, hence respectively $\{0, -3\}$ or $\{0, 5\}$. Neither contains 3.

Three main N already use three lines and cost 12. Four N could only be the main pair plus a balanced pair. If their prefixes differ, they occupy at least three lines. If prefixes agree, two shared lines have different V positions and cost at least 5 each; at total 11 their $\mathrm{F}_5/\mathrm{F}_6$ forms have no actual stars, contradicting mass 1. Thus there are only the two main N. $\mathrm{F}_5 + \mathrm{F}_6$ cannot give $\Gamma = a - 1/2$, so the costs are $\mathrm{F}_4 + \mathrm{F}_7$, with all marginal Γ on the F_7. A

vertex outside the source prefix belongs to at most one main half, proving the entry assertion.

A related local fact will be used with $f \le 12$. Suppose a sole class has value $b = \pm 1/2$ and some outside-prefix B entry equals 3. Its one relevant main half would have marginal $-3/2$. To account explicitly for balanced N: three main N at this budget leave only glue-free F_4 lines. Four N with different prefixes occupy at least three lines, again leaving only F_4 lines. If the main pair and a balanced pair instead share two lines, those lines have different V positions and cost at least 5 each. The two shared lines have N entries (-b,t) and (-b,-t), in distinct positions, with $b = \pm 1/2$ and total cost at most twelve. A target marginal $-3/2$ cannot use F_5 or F_6. If the target line has F_7, its partner must have F_5; the partner law forces $t = b$ after orientation. The target then has opposite coefficients, whose F_7 marginal is $\pm b$, not $-3/2$. If the target has cost at least eight, the pair costs at least thirteen. Thus the shared-pair case is impossible. Five N plus a star and a nonzero glue already exceed 12. Thus the remaining case has just the two main N. It cannot use an extra line: two main lines plus a nonzero-marginal third cost at least 13. Thus it contains a specified main $N = -b$. The line costs at least 7 by the $\mathrm{F}_4/\mathrm{F}_5/\mathrm{F}_6$ laws. The other main line costs at most 5, but would need marginal $\Gamma + 3/2 = b + 1$, which is neither $0, b$ nor $b/2$. Contradiction.

If instead $b = \pm 1$ and $f \le 11$, the same argument leaves other-line cost 4 and marginal 0. Hence $B = 3$ forces $b = -1, f = 11$, and $\mathrm{F}_4 + \mathrm{F}_7$ with F_7 marginal $-3/2$. Here a balanced pair with distinct prefix would need a third line; with equal prefix the only two shared-line costs within 11 are $\mathrm{F}_5/\mathrm{F}_5$ or $\mathrm{F}_5/\mathrm{F}_6$, with no actual stars, contradicting center mass 1. The F_7 therefore contains only its main $N = +1$. If every crossing glue is in $\{0, \pm 1/2\}$, marginal $-3/2$ forces three glues $-1/2$, giving two actual stars and cost 9, not F_7. Thus this last pattern is impossible under that forced crossing grid.

1.4 Two one-loop Z blocks have equal coefficient magnitudes. At $C = 2$ with Z in distinct blocks, each block has one L_3 and $\delta_i = s_i \cdot Z_i/2, s_i$ in $\{+1, -1\}$. Put $\varrho_i = e_{\text{partner}} - e_{\text{attachment}}$. Projection reads

$$Z_1 \cdot (s_1 \cdot c_1 - \varrho_1) + Z_2 \cdot (s_2 \cdot c_2 - \varrho_2) = 0$$

Each parenthesized vector has six entries in $\{+1, -1\}$, three of each: c_i occupies the four coordinates outside the source and ϱ_i its two source coordinates. Nonzero cancellation of two such vectors forces $|Z_1| = |Z_2|$. D does not enter this equation.

1.5 Two half-valued Z blocks with no other flow. If $Z_1 = Z_2 = 1/2$ attach at distinct vertices in distinct blocks and no other block flows, projection forces either both deltas zero, or $|\delta_1| = |\delta_2| = 1/4$. In the zero case the sources coincide and the attachments are opposite endpoints. This is an

exact finite incidence lemma: all 45 blocks and both endpoints are exhausted, and the two arbitrary rational deltas are solved by their nonsingular 2 x2 Gram system and checked in all six coordinates. No δ grid is imposed.

For $|\delta| = 1/4$, the net helper-minus-divergence contribution of one such Z block is $+1/2$ on its source and one side, and zero elsewhere. For equal-source zero flow, F on that source has $H^0 = 0$ at glue-free endpoints and total helper 1, incompatible with w in $\{0, 2\}$.

2. Remove one Z and high helper counts. If $z = 1, d = 0, \Pi \geq 20$. Without cancellation $A \geq 3$ and $C \geq 6$ gives $R \geq 29$. Cancellation requires Z of coefficient $2, \Pi \geq 21$ and $A \geq 2$, again $R \geq 29$. If $z = 1, d = 1, \Pi \geq 20, A \geq 2, C \geq 6$ would force $\Pi = 20$. Both endpoints would then be $f = 8$, with the sole class value $1/2$ at each. The endpoint without Z forces $D = 1/2$, and the other would force $Z = 0$, a contradiction. For $d \geq 2$, the two endpoints each have two classes and Π is already too large. Therefore $z \geq 2$ and $C \geq 2$.

If any endpoint has two classes, $\Pi \geq 21 + h$. Thus $h \geq 5$ exceeds 28 with $A \geq 1, C \geq 2$. If each endpoint has just one nonzero class and $d = 0$, there is at most one Z at each endpoint, so $h \leq 2$; the two helpers may belong to different blocks. If $d > 0$, its source is K and that D fixes the class at both endpoints. The only permitted helpers are then D, Z_u, Z_v in that block, so $h \leq 3$. Thus $h \geq 4$ always forces a multiclass endpoint. At $h = 4$ the only possible budget is $\Pi = 25, A = 1, C = 2$. $A = 1$ requires K to cross two M-edges and both M-Z values 2. The other two helpers lie outside these blocks and create a new nonzero class at some endpoint after normalization. One endpoint therefore costs 13, the other at least 11, giving $\Pi \geq 26$. Thus $h \leq 3$.

At $h = 3$, if both endpoints have one class, all helpers are $D + Z_u + Z_v$ in one block. $\Pi \geq 21, A \geq 2$ gives $C \leq 5$. The one-block lemma makes the Z equal with zero flow, and the exact kernel deletes both Z while modifying the existing D and adding at most one E. Support decreases, contradicting lower bound 31. If an endpoint has multiple classes, $\Pi \geq 24$. $A = 1$ is impossible: two M cancellations use Z of coefficient 2 at both crossing endpoints and the third helper creates an outside nonzero array. Hence only $\Pi = 24, A = 2, C = 2, X = 0$ remains, with endpoint cost total 22.

2.1 $h = 3$, three Z. Three Z need at most two blocks because $C = 2$. A zero $-\beta$ outside block would be an opposite Z pair needing two loops besides the remaining Z block's loop. Thus all occupied Z blocks have nonzero β. $A = 2$ requires two sole M-Z of coefficient 2 cancellations plus the third Z outside, or would require an impossible outside zero β. In the former case one endpoint has two classes and the other a value 2 class, costing at least $13 + 11 > 22$. This excludes three Z.

2.2 $h = 3$, one D and two Z. Both Z in one block either share D, already excluded by the kernel, or give two classes at both endpoints, exceeding the budget. Both Z in distinct M-blocks are excluded by projection with $C = 2$.

Both outside leave too many arrays for $A = 2$. Thus one Z is in M and one outside.

If D shares the M-Z block, its other two M-arrays plus the outside array require $A = 3$. A different M-block for D is impossible with two attachments: its source would be K, and the M-edge at either of its endpoints is that same block. Therefore D is outside M and must share the outside Z block with $\beta = 0$. The M Z is 2 and the outside Z is $-2D$. item 1.4 gives $|D| = 1$. At the shared M-Z endpoint the class values are 2 and ± 1; item 1.1 makes its cost at least 14. The other endpoint has a class of absolute value 1 and costs at least 9. Their total exceeds 22.

Thus only $d = 0, z = 2$ remains.

3. Two Z in one block, or at one attachment. For a common block, $\Pi \geq 20$. If $A \geq 3, C \leq 5$ immediately. $A = 2$ would require $\beta = 2$ in M and endpoint joint cost at least 18, giving $\Pi \geq 21$ and again $C \leq 5$. The one-block lemma makes $Z_u = Z_v = a$ and flow zero. Source balance and its purely local E23 endpoint argument give joint cost at least 23, hence $\Pi \geq 24$. An outside block has $A \geq 4$, and a nonunit block in M has $A \geq 3$; both exceed 28 with $C \geq 2$. If $a = 1$ in M, the local U25 endpoint bound gives $\Pi \geq 25$, also too large with $A = 2, C = 2$. The E23 / U25 arguments use the common class value and source balance; they do not require that it was produced by a D instead of equal Z.

Now put both Z at one attachment but in distinct blocks. That center costs 13 and the other, helper-free active center 9, so $\Pi \geq 23$. At most one Z block is in M, giving $A \geq 3$. Only $\Pi = 23, A = 3, C = 2, X = 0$ remains. The M Z must be 2 to cancel its array; item 1.4 forces the other Z to be ± 2. The active center is $f = 13$, excluded by item 1.2.

It remains two distinct blocks and two distinct attachments, each with one nonzero class. $\Pi \geq 20$.

4. Both Z blocks outside M. $A \geq 5$ gives $\Pi \leq 21, C \leq 3$. If $\Pi = 20$, both endpoints are $f = 8$ with $Z = 1/2$ and all inactive $f = 5$. No Z-free block can flow. item 1.5 gives either the impossible equal-source zero-flow case, or deltas of magnitude $1/4$. Each outside group contains at most one M-edge. The two net helper contributions therefore reach at most two M-edges, each by $1/2$. At an untouched M-edge, F would require $H^0 = 2$, while $H^0 \leq 1$. This also covers the third-loop and mixed-column slack.

If $\Pi = 21, C = 2$ is forced. item 1.4 gives equal absolute class values. Endpoint total cost is at most 18 and each endpoint $f \leq 10$. The low-cost class list leaves either both values 1, or values of absolute value $1/2$ with at most one negative endpoint. An absolute-value 1 negative class already costs 10, exceeding 18 when paired with the positive class cost 9. Inactive vertices have $f \leq 7$ and B entries zero / one.

For values 1 the active B entries are nonpositive. A value $1/2$ center at $f \leq 10$ has all marginals zero. A value $-1/2$ center costs 10 and has only its two main lines, each F_5 of marginal $-1/2$, so its B entries are at most

2. It is the only possible such endpoint. Consequently every H^0 entry is at most 3/2, including its edge to the other glue-free active center. Each outside one-loop Z contributes to at most one M-edge, leaving an untouched M-edge that would require $H^0 = 2$. Contradiction.

5. Both Z blocks in M. They occupy distinct M-edges U,V; call the unaffected M-edge W. Projection requires an outside-M flow block, costing two more loops, so $C \geq 4$. Every such outside flow vector must be constant on W. The proved exact incidence says its matching contains W and neither U nor V; it therefore has no pair-node divergence at U or V.

With no array cancellation, $\Pi \geq 20, A \geq 3$. The possible budgets have $\Pi = 20$ or 21 and $C = 4$ or 5. With one cancellation, Z of coefficient 2 gives $\Pi \geq 22$ and only $\Pi = 22, A = 2, C = 4$ remains. With two cancellations, only $\Pi = 23, A = 1, C = 4$ remains and both Z are 2. These include all mixed / extra-A slack.

5.1 $C = 5$. Only $\Pi = 20$ without cancellation fits. Both Z are 1/2 at glue-free $f = 8$ endpoints, with inactive $f = 5$. There is exactly one outside flow block. At either affected M-source, $H^0 \leq 1/2$ and helper weight 3/4, so its divergence is at most $-3/4$. Only the other M-Z block can provide that divergence. A one-loop $Z = 1/2$ block has $|\delta| = 1/4$, so each of the two Z blocks would need at least two loops, in addition to the outside block's two. This requires $C \geq 6$, not 5.

5.2 $C = 4$ *without cancellation.* Each Z block has one L_3. No endpoint can have a separator, since $f \leq 10$, and inactive $B \leq 1$. At source U, with Z values a,b,

$$w_U \leq (1 - a) + a + b/2 + |b|/2 = 1 + \max\,(b, 0)$$

Thus $b \geq 1$, and similarly $a \geq 1$. The low-cost class list forces $a = b = 1, \Pi = 21$, active $f = 9$ and all inactive $f = 5$. F at U,V then forces divergence $-1/2$ at each and B from each inactive mate to its active mate equal 1. The two M-flows contribute $+1$ at W. Projection on W forces the outside flow to contribute -1 there and $+1$ at another pair-node e. At e, F requires $H^0_e = 1$. An edge incident to an active $f = 9$ center has $H^0 \leq 1/2$, so e must join the two inactive mates p,q of the Z attachments u,v.

But the private $f = 9$ lines force p,q into opposite halves. To see this, the u line contains $U = \{u, p\}$ and a third vertex t. It is dense. If $t = q$, its glue to v lies on a forbidden separating v line. If $t = v$, a positive u-glue forces an inactive opposite $f = 5$ line to be dense, which activates the likewise forbidden separating v line. Thus t belongs to W. The opposite half then contains $V = \{v, q\}$ and the other W endpoint. Its v line is dense and forces all lines in that partition to occur. Hence p,q lie in opposite halves and $H^0_{pq} = 0$, not 1.

5.3 One or two cancellations. With one cancellation $a = 2, \Pi = 22$ leaves the other endpoint $f \leq 9$. The preceding source inequality still forces $b \geq 1$, so $b = 1$. The exact endpoint costs are $11, 9$ and all inactive $f = 5$. At the unit source V, H_V^0 is $-1/2$ or 0 and helper weight is 2, so divergence is $-1/2$ or 0. But its only contributing flow is the one-loop Z of coefficient 2 block, whose divergence there has magnitude 1. Contradiction.

With two cancellations $a = b = 2$, both endpoints are $f = 11$ and all inactive $f = 5$. At either source H^0 is $-3/2$ or -1 and helper weight 3, so divergence is again $-1/2$ or 0, whereas the other one-loop block gives magnitude 1. This completes both-in-M placements.

6. One Z block in M and one outside. Let $M - Z = a$ attach at u with source $e = \{u, p\}$; let outside $Z = b$ attach at v. First suppose $C \leq 3$, so no Z-free block flows. If neither endpoint has a separator, classify e in the outside group:

- If e is its source, $p = v$ and F gives $w_e = (1 - a - b) + (a + b) = 1$, impossible.
- If e is a side, both flow vectors are constant on e and the outside source is disjoint from it, contradicting the antisymmetric M-Z term in projection.
- If e is absent, F requires $B_{pu} = 3$, since $B_{up} = 1 - 2a$ and e has no outside helper contribution or divergence.

The following budgets ensure the stated no-separator premise and exclude the final $B_{pu} = 3$ possibility.

6.1 $A = 3$. The M array must cancel, so $a = 2, \Pi \geq 22$, and $\Pi \leq 23$. At $\Pi = 22, f(u) \leq 12, f(v) \leq 9$, and every inactive center is light. If p is inactive its B entry is at most 1; if $p = v$, the proved outside-prefix $f \leq 10$ bound gives $B \leq 2$. At $\Pi = 23$ the budget forces $C = 2$. item 1.4 gives $|b| = 2$, making both active costs 11 and all inactive $f = 5$. There are still no separators. If $p = v$, item 1.3 gives B_{vu} in $\{0, -3\}$ or $\{0, 5\}$, not 3. This includes $\Pi = 22, C = 3$ and all mixed slack.

6.2 $A \geq 4$ and $C \leq 3$. $\Pi \leq 22$. If $\Pi \leq 21$, both active $f \leq 10$ and inactive $f \leq 7$, so the same outside-prefix bound excludes 3. At $\Pi = 22$, only $A = 4, C = 2$ fits. Both active $f \leq 12$, all inactive are light, and item 1.4 gives $|a| = |b|$. The only remaining possibility is $p = v$ with $f(v) > 10$. Then $f(u) \leq 9$:

- $f(u) = 8$ forces $a = 1/2, b = \pm 1/2, f(v) \leq 12$, excluded by item 1.3.
- $f(u) = 9$ forces $a = 1, b = \pm 1, f(v) \leq 11$. item 1.3 leaves only $b = -1, f(v) = 11$, an F_7 main line of marginal $-3/2$. All inactive centers are $f = 5$. That line's half contains both active vertices u,v and its outside source partner, so all opposite centers are inactive $f = 5$. Their forced half-grid makes this F_7 impossible by item 1.3.

6.3 $C = 4$. Only $\Pi = 20, A = 4, C = 4, X = 0$ fits. Both Z are $1/2$ at glue-free $f = 8$ endpoints and all inactive $f = 5$; hence H^0 is nonnegative and at most 1 everywhere. If no Z-free block flows, item 1.5 gives the already impossible equal-source zero-flow case, or each Z contributes net $1/2$ at its source and one side. If a Z-free block flows, it uses two loops, leaving one L_3 at each Z block and giving the same net contributions.

The M Z contributes to two M pair-nodes; the outside Z contributes to at most one. Their total contribution on M is at most $3/2$, and each M node receives at most 1. Thus $H^0 +$ net $_{\text{helper}} - 2$ is nonpositive on M and strictly negative at at least two M nodes. It is nonnegative off M. With no further flow this is impossible. With the one Z-free flow, it is also impossible: a single graph edge has only one negative divergence node. This treats all idle-loop placements in the $C = 4$ budget.

These cases exhaust two Z in different blocks and at different attachments. Together with S14.6, items 2–3, they close every $m = 2Z$-present matching $R = 28$ case.

S14.7. **Three or four centers with physical helpers.**
Scope. Matching $R = 28$, $Z > 0$, three or four active centers.
Obstruction. Cancellation and endpoint costs restrict (Π, A, C, h); physical projection and private cost exclude the remaining two- and three-helper placements.
Conclusion. No such identity; each zero-flow use is justified by its loop budget.

Assume matching corrections, $R = 28, Z$ present and m in $\{3, 4\}$.

Use the lower bound 31 identities, normalization and local costs. Set $h = d + z, R = \Pi + A + C + X, 2\Pi = \sum f + 2h$. Inactive $f \geq 5$ with no $f = 6$; active $f \geq 8$, or 9 if $\Gamma \neq 0$. Two nonzero classes give $f \geq 13$. An active $f = 8$ has exactly one nonzero class of value $1/2$. At $f = 9$, a nonzero class has value 1; otherwise all helpers vanish. A sole value 2 class costs at least 11; if another class is present $f \geq 13$, so the bound 11 remains valid. The helper-array charge is $\Pi \geq 21 + h$ for $m = 4$ and $\Pi \geq 20 + h$ for $m = 3$. Matching rigidity gives $\Omega = -2e_M$.

Every Z block needs a loop. Exactly one global Z needs $C \geq 6$. With Z present, $C = 1$ is excluded by the proved support-decreasing normalization against lower 31, so $C \geq 2$. A Z-free nonzero-flow block costs two loops. Mixed columns vanish in all equations used here.

1. Cancellation costs and projection facts. If an M-array cancels, its block has $\beta = \zeta_u + \zeta_v = 2$. If only one source endpoint is active, it has class value 2 and costs $f \geq 11$. If both are active, their joint cost is at least 18. Indeed $f \leq 9$ permits the relevant class values only $0, 1/2, 1$, with values 1 costing 9, and $f = 8 + f = 10$ cannot sum to $\beta = 2$: the full $f \leq 10$ single-class list is $\{1/2, 1, -1/2, -1\}$, while a second class already costs 13. At joint cost 18, the only cancellation pattern is costs $9, 9$ and both relevant values 1. All these statements allow balanced N and additional classes.

For flow blocks with group M, the vector $c = 1_W - 1_V$ is constant on each M-edge. A lone endpoint Z on an M-edge gives a nonzero antisymmetric

component there in the projection equation. Consequently all flow confined to M cannot coexist with any unpaired endpoint Z in an M-block. If two distinct Z blocks use $C = 2$, every loop is in them. If $C \leq 3$, no additional Z-free nonzero-flow block can occur.

The proved one-block lemma says that if all Z form the endpoint pair of one block and $C \leq 5$, the two Z are equal and all flow is zero. If a D already occupies that same block, the exact move

$$(D, Z_u, Z_v, E) \to (D + t, Z_u - t, Z_v - t, E + t), t = Z_u = Z_v,$$

removes both Z, adds at most one E and changes an already present D. It strictly decreases support while preserving all classes and the correction terms, contradicting the proved global lower 31 at width 31.

2. Four attachments. $\Pi \geq 23$ excludes $z = 1$ immediately, since $C \geq 6$. Hence $z \geq 2, C \geq 2$.

2.1 $A = 0$. If K consists of two full M-edges, the third array cannot cancel. Otherwise K contains one full edge U and one endpoint of each other edge. Canceling the two crossing arrays needs two class-value 2 vertices, costing at least 11 each. The other two active vertices cost at least 8 each and the inactive vertices at least 10 together. At least three helpers are needed to cancel all three arrays, so

$$\Pi \geq \text{ceil } ((11 + 11 + 8 + 8 + 10)/2) + h = 24 + h \geq 27$$

With $C \geq 2$ this exceeds 28. Thus $A \geq 1$ and $\Pi \leq 25$, whence $h \leq 4$.

2.2 $A = 1$. At least two M-arrays cancel. If $h = 4, \Pi = 25$ is forced, and every active vertex is $f = 8$ and every inactive $f = 5$. All relevant class values are $1/2$, so no block has $\beta = 2$. If $h = 2$, the helpers are two Z of coefficient 2 needed in two canceled blocks. At distinct endpoints the active cost is at least $11 + 11 + 9 + 9 = 40$; at one common endpoint it is at least $13 + 9 + 9 + 9 = 40$. Thus $\Pi \geq 27$, impossible.

For $h = 3$ there are $z = 3$ or $d = 1, z = 2$. With $z = 3$, at least one canceled block has a sole Z of coefficient 2, costing 11 at its endpoint, and at least one active vertex is helper-free and costs 9. The remaining two cost at least 8, giving $\Pi \geq \text{ceil } ((11 + 9 + 8 + 8 + 10)/2) + 3 = 26$.

For $d = 1, z = 2$, distribute the three helpers among the two canceled blocks. If one canceled block uses D plus Z, the other has a sole Z of coefficient 2. At most three active vertices carry nonzero classes, so the same $11, 9, 8, 8$ bound gives $\Pi \geq 26$. If one canceled block uses a Z pair and the other a sole unit D, disjoint source endpoints cost at least $18 + 18 = 36$; intersecting sources give a two-class vertex and a helper-free vertex, which only increases that bound. Again $\Pi \geq 26$.

It remains two cancellations each using one helper, with the third elsewhere. Sole unit D and sole Z of coefficient 2 require active cost at least 37

if their endpoints are distinct, or at least $13+9+8+8=38$ if they overlap. Two sole Z of coefficient 2 require cost at least 38 at distinct endpoints; if they share an endpoint, that vertex costs 13 and a helper-free vertex costs 9, with the other two at least 8 each, again 38. Thus no $h=3$ case fits $\Pi=25$. These are all allocations; an outside zero $-\beta$ block needs two helpers in addition to the two cancellations and is covered by the $h=4$ exclusion.

Therefore $A\geq 2, \Pi\leq 24$ and $h\leq 3$.

2.3 The last $h=2, h=3$ budgets. At $h=2$ the helpers are two Z. If both attach at one center, the active cost is at least $13+9+9+9$, already too large. Otherwise the other two active centers cost 9 each, so $\Pi\geq$ ceil $((8+8+9+9+10)/2)+2=24$. The only budget is $\Pi=24, A=2, C=2, X=0$. Its two Z endpoints are $f=8$ with values $1/2$; no M-array cancels, contrary to $A=2$.

At $h=3, z=3$, at least one active center is helper-free, giving $\Pi\geq$ ceil $((8+8+8+9+10)/2)+3=25$. At $h=3, d=1, z=2, \Pi=24$ forces all four active $f=8$ and inactive $f=5$. Every relevant class is $1/2$, so $\beta\leq 1$ and again no array cancels. This completes $m=4$ with Z present.

3. Three attachments: remove small A and large h. A single Z has $\Pi\geq 22$: with $d=0$ the other two active vertices cost 9 each; with $d\geq 1$ use $\Pi\geq 20+h$. At least one M-array remains untouched, so $A\geq 1$. Together with $C\geq 6$ this gives $R\geq 29$. Thus $z\geq 2, C\geq 2$.

3.1 $A=0$. K must meet every M-edge once. Each canceled array then needs Z of coefficient 2 at its active endpoint. Active cost is at least 33, inactive cost 15 and $h\geq 3$, so $\Pi\geq 24+h\geq 27$. Hence $A\geq 1, \Pi\leq 25$ and $h\leq 5$.

3.2 $A=1$. If K meets all three M-edges once, two distinct vertices have Z of coefficient 2. At $h\geq 3$, costs $11, 11, 8$ and inactive 15 give $\Pi\geq 26$. At $h=2$ the third active vertex is helper-free; only $\Pi=25, A=1, C=2, X=0$ remains. Both Z are lone endpoints in distinct M-blocks, contradicting projection.

Otherwise $K=U$ union $\{w\}$, where U is an M-edge and w lies on another M-edge V. The fully inactive third edge consumes the one alternating. Both U and V cancel; V has its sole $Z_w=2$ and $f(w)\geq 11$. At $h=2, z\geq 2$ forces a sole Z of coefficient 2 at an endpoint of U, with the remaining active vertex helper-free. This again gives $\Pi=25, A=1, C=2$ and the projection contradiction.

At $h=3$, all helpers lie in the two canceled M-blocks: an outside nonzero β would require another alternating, and an outside zero β would need two extra helpers. V contains only Z_w. Cancellation at U costs joint $f\geq 18$, so $\Pi\geq$ ceil $((18+11+15)/2)+3=25$. Equality requires both U-class values 1 with endpoint costs $9, 9$. $D+Z$ would make the two endpoint values unequal unless the Z were zero. Thus U has $Z=1, 1$, and V its lone Z of coefficient 2. $C=2$ confines loops to M, again impossible.

At $h = 4, \Pi \geq$ ceil $((8+8+11+15)/2)+4 = 25$. Equality forces the two U endpoints $f = 8$, so their relevant values sum to at most 1, not $\beta = 2$. $h \geq 5$ is already too costly. Consequently $A \geq 2, \Pi \leq 24$ and $h \leq 4$.

If $h = 4$, the line budget is at most 40, with active total at most 25. $A = 2$ requires a cancellation. A cancellation within K costs 18 plus the third active cost 8; a crossing cancellation costs $11 + 8 + 8$. Both exceed 25. Thus $h = 4$ is impossible, and only $h = 2$ or $h = 3$ remain.

4. Three attachments, $h = 2$: two Z and no D. Two Z at one attachment cost at least 13 there,9 at each other active center and 15 inactive, giving $\Pi \geq 25$. Therefore their attachments are distinct; the third active vertex is helper-free. $\Pi \geq 22$.

4.1 Both Z in one block. $A \geq 2$ implies $C \leq 4$. The one-block lemma gives equal Z and zero flow. Source balance requires a nonzero separator at one endpoint. Endpoint costs at least $13, 8$, third active cost 9 and inactive 15 give $\Pi \geq$ ceil $((13+8+9+15)/2)+2 = 25$, impossible.

4.2 Distinct blocks, both in M or both outside M. Both outside requires $A \geq 5$ and $R \geq 22+5+2 = 29$. Both in M requires an additional Z-free flow block by projection, so $C \geq 4$. Without array cancellation, $A \geq 3$ already gives 29. One canceled array has a Z of coefficient 2 and forces $\Pi \geq$ ceil $((11+8+9+15)/2)+2 = 24$; with $A \geq 2$ this gives 30. Two canceled arrays force $\Pi \geq 25$ and $A \geq 1$, again giving 30.

4.3 One $MZ = a$ at u, one outside $Z = b$ at v. If $A = 3$, cancellation forces the matching Z to have $a = 2$. The active costs are at least $11, 8, 9$ and the inactive cost is 15, so $\Pi \geq 24$ and $R \geq 24+3+2 = 29$, outside the present bound. Thus $A \geq 4$. The only remaining budget is $\Pi = 22, A = 4, C = 2, X = 0$; both Z endpoints have $f = 8$ and $a = b = 1/2$, the helper-free active center has $f = 9$, and the inactive centers have $f = 5$.

Write $e = \{u, p\}$ for the M source. The M endpoint has no nonzero separator, so $B_{up} = 1-2a$. All flow is confined to the two Z blocks. Classify e relative to the outside matching group:

- If e is its source, $p = v$. Neither endpoint has a separator, so $H^0_e = 1-a-b$. Helpers contribute $a+b$ and divergence is zero, giving $w_e = 1$ rather than 2.
- If e is a side, both flow vectors are constant on e and the outside source is disjoint from e. The M Z gives a nonzero antisymmetric projection on e, a contradiction.
- If e is absent, it has no divergence or outside helper contribution. The F equation forces $B_{pu} = 3$. An inactive p has $B \leq 1$. At the outside Z center, u is outside its prefix and the proved $f \leq 10$ bound gives $B \leq 2$. At the helper-free center, the new B_{10} forms give $B \leq 2$: at $f = 9$ entries are $0/1$; at $f = 10$ they belong to $\{0, 2, -1, 1\}$ because the two marginals are -1 and $1/2$. None can equal 3.

This includes the extra helper-free attachment without assuming its B entries are always nonnegative. It closes $h = 2$.

5. Three attachments, $h = 3$. $\Pi \geq 23$. At $\Pi = 23$ the active line total is 25, so there are two $f = 8$ vertices and one $f = 9$, with all inactive $f = 5$. No vertex has two nonzero classes. Every nonzero class is $1/2$ or 1 and no block can have $\beta = 2$; thus $A \geq 3$ and the budget is $\Pi = 23, A = 3, C = 2, X = 0$. Every nonzero helper block has positive β, so all helpers must lie in M.

With three Z, they occupy at least two blocks and some block has a lone endpoint Z. $C = 2$ confines flow to M and projection excludes it. With one D and two Z, the D is on the sole full M-edge in K. The Z either both lie in its block, excluded by the equal-Z kernel reduction of item 1, or one lies at the third active vertex in another M-block. The latter is a lone Z, excluded by projection with $C = 2$. Thus $\Pi = 23$ fails.

At $\Pi = 24$ the exact budget is $A = 2, C = 2, X = 0$ and at least one M-array cancels.

5.1 Three Z. There are at most two Z blocks. A zero $-\beta$ outside block would contain an opposite Z pair and need two loops, in addition to the remaining Z block's loop. Thus every Z block has nonzero β. An outside Z block plus at most one canceled M-array would require $A \geq 3$. The remaining possibility of two canceled matching arrays and an outside Z must also be excluded. Two value-two matching Z at different attachments, together with an outside Z, cost at least $11+11+8+15$ in lines, hence $\Pi \geq 26$ after three helpers. If the outside Z shares an attachment, the two-class center costs at least thirteen and the line charge is at least $13 + 11 + 9 + 15$, hence $\Pi \geq 27$. Both exceed $\Pi = 24$. All Z therefore lie in M, with a lone endpoint Z in some block, contradicting projection with $C = 2$.

5.2 One D and two Z in one block. The one-block lemma gives equal Z and zero flow. If D is in that same block, the kernel reduction contradicts lower 31. If D is in a different block, both blocks have nonzero β. $A = 2$ forces both into M; their disjoint source pairs would require four attachments, not three. This case is impossible.

5.3 One D and two Z in distinct blocks. Both Z in M are excluded by projection with $C = 2$. Both outside cannot fit $A = 2 : a$ single D can cancel at most one M-array or one outside β, leaving too many arrays. Thus exactly one Z is in M and one outside.

If D shares the M-Z block, the other two M-arrays and the outside Z array give $A \geq 3$. If D is in a different M-block, $A = 2$ requires $D = 1$ and $M - Z = 2$. Their endpoints exhaust K. The outside Z adds a second class somewhere. At a D endpoint the active costs are at least $13, 9, 11$; at the value 2 endpoint they are at least $9, 9, 13$. Either exceeds the $\Pi = 24$ active budget 27.

If D is outside M, the M Z must be 2, and the outside β must vanish. Thus D and the outside Z share a block, $Z = -2D$, and its endpoint classes

are nonzero opposites. If its source contains the M-Z endpoint, that center has two classes and $f \geq 13$, the other source endpoint costs at least 8, and the third active vertex is helper-free with $f \geq 9$. This gives $\Pi \geq 26$. Otherwise the M-Z endpoint costs 11 and the opposite classes cost at least $8 + 9$, giving $\Pi \geq 25$. Neither fits $\Pi = 24$.

This exhausts $m = 3$ with Z present. All uses of $C = 0$ or confined flow have been justified by a loop budget or the one-block lemma; zero flow alone has never been used to discard idle loops.

S14.8. **One active center with physical helpers.**
Scope. Matching $R = 28$, one active center, nonzero helpers.
Obstruction. Derive the retained-28 tuple list. Sole-Z source flow and local class costs exclude its one- and two-Z branches.
Conclusion. Every tuple is excluded, allowing balanced N and arbitrary actual-star positions.

1. Claim, scope and inputs. In the canonical rational PM2 $(6, 2)$ model, a normalized identity with matching corrections, $R = 28$, exactly one N attachment vertex u, and at least one nonzero normalized helper is impossible. All signs, balanced N at u, mixed support X and idle loops are allowed. Here $m = 1$ means the N attachment count in the attachment partition; balanced N at other vertices would give $m > 1$ and is outside this claim.

The preceding lower-bound 31 component is the preliminary input.

The proved $R = 27$ exclusion is not being extrapolated. item 4 derives every $R = 28$support tuple, and S14.8, items 5–6 prove the new tight cases.

Use scale 3, actual center star masses 1, (S4.8)–(S4.9) and the marginal law (S4.2). The line cost is $F = g + 2s + 2n$. Matching rigidity O29, valid at $R \leq 29$, gives $\Omega = -2e_M$ and $w = 2M$. Normalization excludes D at $m = 1$. Every Z is at u in a distinct block and a distinct nonzero N class. Each class has four possible N on four distinct private lines. Its two h-side sums are -Z; two N have coefficients -Z,-Z, one per side. A balanced class has even N support, and a balanced pair has opposite coefficients on distinct lines.

Equations (S4.6) and (S4.7) are denoted (F) and (P) in this component. We use B, H^0 as defined in (S4.5), with $c_B = \mathbf{1}_W - \mathbf{1}_V$. For $m = 1$ each β equals its Z. Mixed columns have zero contribution to these equations and to V/h, center and block equations.

Proved local facts include:

- Inactive $f \geq 5$; strict excess is at least 2. Inactive $f \leq 9$ has a single marginal $-1/2$ and every B entry is zero or one.
- Active $f \geq 8$, or $f \geq 9$ if $\Gamma \neq 0$. Two nonzero classes require $f \geq 13$.
- Actual center mass requires a nonzero star. Nonzero Γ requires a glue. Every N-containing line costs at least 4, and F_4 is glue-free.
- A one $-NF_5$ line has marginal -n or $-n/2$; all its nonzero glues have magnitude equal to the absolute marginal. A one $-NF_6$ has marginal 0.

- Two opposite N on a line of cost ≤ 7 give F_4(same cell, no glue or stars $), \mathrm{F}_6$(marginal 0), or F_7(marginal plus or minus their amplitude).
- Same-prefix classes from different groups occupy different V positions on each common line. Different prefixes share exactly one possible private line, the half containing u and both prefix mates.
- A nonzero class and a nonzero separating marginal require $f \geq 13$.
- Every Z block needs a loop; a Z-free flowing block needs at least two loops. A sole global Z costs $C \geq 6$. If its block has only one loop, the stronger argument gives $C \geq 7$(proved S13.5, part G, L1 +).

2. universal single $-Z, C = 6$ source-flow lemma. **SZ6.** In any normalized identity with exactly one global Z and $C = 6$, coordinate flow divergence at its source edge is zero. D, N, correction weights and mixed columns may be arbitrary. This lemma only uses (P) and the proved loop costs.

Let the Z be $a \neq 0$ at u, source $U = \{u, t\}$, in block B_0. Its block needs at least two loops, since one would force $C \geq 7$. The projected residual $\varrho - q_0 \cdot c_0$ has support two if $q_0 = 0$, and six if $q_0 \neq 0$, where $q_i = 2\delta_i/a$ and $\varrho = e_t - e_u$. One other four-supported block vector cannot supply either residual. Thus at least two other flowing blocks are needed, each Z-free and costing two loops. $C = 6$ forces precisely these two blocks $\mathrm{B}_1, \mathrm{B}_2$, two loops in each, and two loops in B_0.

Suppose B_1 carries nonzero flow on U. Then U is one of its side pairs, so $c_1(u) = c_1(t) = \varepsilon$ in $\{+1, -1\}$, and $q_1 \neq 0$. B_0 has zero divergence on U. From the two coordinates u,t of

$$q_0 \cdot c_0 + q_1 \cdot c_1 + q_2 \cdot c_2 = \varrho,$$

c_2 must have different values at u,t. If these are $-1, +1$, the equations force $q_1 = 0$, contradiction. Otherwise one endpoint value is zero and the other is plus or minus one. They force $|q_2| = 2$ and $|q_1| = 1$.

The source of B_2 therefore contains one of u,t and an outside vertex x. At $x, c_2(x) = 0$ and $c_0(x) = \pm 1$. The equation gives $|q_0|$ either 0 or 1, because $c_1(x)$ is zero or ± 1. If $q_0 = 0$, the other three outside coordinates with nonzero c_2 cannot cancel a term of magnitude 2 using $q_1 \cdot c_1$, whose magnitude is at most one. If $|q_0| = 1$, cancellation at all three of them requires c_1 to be nonzero at all three. But c_1 already has two nonzero coordinates u,t, and has only four in total. Again a contradiction. Therefore no flowing outside block has U as a side. The source edge's flow divergence is zero, proving SZ6.

The exact checker tests all 258 fixed normalized projection configurations with a proposed source-side flow block. Their rational affine systems are empty. Relabeling covers every source / block and both Z endpoints. No three-block network support search is involved.

3. Three collision-safe local controls.

3.1 A value 2 *single class cannot fit* $f \leq 13$ *on the half-grid.* **Z13.** Suppose u has exactly one nonzero class of value 2, all glues at u are in $\{0, +1/2, -1/2\}$, and $f(u) \leq 13$. Balanced N may be present. Then no local $V/h/$ center solution exists.

Here $\Gamma = 3/2$, so at least three glues are necessary. Five N plus a star would cost at least $10 + 3 + 2 = 15$. With three main N, there are three distinct N-lines; total cost ≤ 13 permits only $\mathrm{F}_4, \mathrm{F}_4, \mathrm{F}_5$. Its sole marginal is $\pm 1/2$ by the F_5 glue-magnitude rule, contradicting Γ. Four main N require four distinct lines and cost at least 16.

With precisely two main $N = -2$ on distinct lines, a nonzero marginal on a main line costs at least 7: an F_5 marginal would be 1 or 2 and its glues violate the half-grid, while F_6 has marginal 0. Two such lines would exceed 13. A nonzero N-free extra line must be F_5 after the two F_4 main lines; its marginal has absolute value $1/2$, not $3/2$. The remaining possibility is one main line with marginal $3/2$, necessarily three glues $+1/2$. Its actual star vector is $(1, -1, -1)$ up to permutation, so that line costs 11 and its partner at least 4, again exceeding 13.

The only other four-N possibility is two main $N = -2$ and one balanced pair $+t, -t$ on distinct lines. Its cost forces exactly three glues $+1/2$ and one actual star of coefficient 1. Effective stars l-r are in $[-1, 0]$, so the aggregate N vector l-r-S is in $[-2, 0]$. The positive balanced N must collide with one main -2 at the same V cell. The other main -2 must occupy the actual-star cell with effective value -1, requiring two $+1/2$ glues on that main line. The remaining negative balanced N is away from that star and requires $t \leq 1$. The first main residual $-2 + t$ is also away from the star, forcing $t \geq 1$. Hence $t = 1$, and it too requires two $+1/2$ glues on its distinct main line. At least four glues would be needed, contradiction.

3.2 Two nonzero classes at $f = 13$ *cannot have value* 2. **T13.** At a center with two distinct nonzero classes and $f = 13$, neither class has value 2. Balanced N are permitted but cannot occur at this cost.

Six N plus a star cost at least 14. Five N would distribute as $3 + 2$: different prefixes need at least four lines costing 16; equal prefixes have at least three lines with distribution $(2, 2, 1)$, costing at least $5 + 5 + 4 = 14$ because shared N positions differ. Thus exactly four N remain, two per class. No additional N-free line fits. Four distinct N-lines cost 16, so only one or two common lines need be considered.

With one common line, the other two lines cost at least 4 each. At cost 13, either the common line is F_4 and the other pair is $\mathrm{F}_4 + \mathrm{F}_5$, or the common line is F_5 and both others F_4. In the first, cancellation forces opposite class values, total $\Gamma = -1/2$, and the singleton F_5 marginal is $-1/2$. Its class values have absolute value $1/2$ or 1. In the second, the common F_5 forces equal values and has no actual star; the singleton stars sum to one, forcing both values $1/2$.

Two common lines require equal prefixes and different N positions. The proved $F_5/F_6/F_7$ collision rules leave only $F_5 + F_8$ or $F_6 + F_7$ at cost 13. In $F_5 + F_8$ the values are equal. F_8 with no glue has two stars and forces value $1/2$. For F_8 with two glues and one star, call the equal class value v. The F_5 marginal is v, and center mass 1 gives the F_8 marginal $r_8 = v - 1/2$. If its star is at an N position, its zero glue is at the third position and $r_8 = 0$, so $v = 1/2$. If its star is at that third position, its zero glue gives $r_8 + 1 = v + 1/2 = 0$, so $v = -1/2$. In $F_6 + F_7$ the values are opposite; F_6 has marginal $0, F_7$ has marginal plus or minus the class value, and total marginal $-1/2$ forces absolute value $1/2$. No case has value 2.

The checker independently imposes first class value 2 and center mass 1 on every rational $F \le 13$ glue / star mask, for the two-common-line and both one-common-line position types. All 3417 systems with faithful nonzero masks are impossible. No bound on the second coefficient is imposed. This lemma does not assert that every $f = 13B$ row is zero / one: equal negative class values can produce two negative marginals.

3.3 Opposite different-source classes at cost ≤ 15. **O15.** Suppose the two Z are at u with different sources $\{u, t_1\}, \{u, t_2\}$, values a,-a, one source equal to the M-edge at u, every inactive B row is zero / one, and flow is confined to these two blocks. If $f(u) \le 15$, there is no identity.

Write $t_1 = u^*$, the matching mate. Neither helper source can be a side of either block, since both sources contain u. Thus (F) requires

$$H^0_{u,u^*} = 2 - a, H^0_{u,t_2} = a.(E)$$

$\Gamma = -1/2$. Five N require four lines and cost at least 16; seven N plus a glue and a star cost at least 17. Six N therefore mean two main N per class plus a balanced pair: other $3 + 3$ or $4 + 2$ main distributions would need five lines costing 20. Their cost 15 forces one glue $-1/2$ and one actual star 1. The active B row is zero / one, so (E) forces $a = 1$ and both entries one. On the unique glueline the effective star vector is a permutation of $(0, 1/2, 1/2)$. The actual star is either absent there or is a unit vector there. The required aggregate N therefore has exactly two half-integer coordinates. The four main N coefficients are integers. The sole balanced pair lies on distinct private lines and can occupy at most one cell of this glueline. One arbitrary rational entry cannot supply both half-integer coordinates. This contradiction permits the sole actual star at every position.

With four N there are exactly three N-lines, a common line and two singleton lines, and no extra nonempty N-free line. The common line costs at most 7. If F_7, its sole marginal is $\pm a = -1/2$; the zero / one active B row forces $a = 1$ via (E), a contradiction. If F_6, the singleton pair is $F_4 + F_5$; its unique marginal $-1/2$ is not on the common line. Equation (E) again forces $a = 1$ and the unique half to contain both mates, contradiction. If common F_4, every singleton cost pair at total ≤ 11 that can satisfy $\Gamma = -1/2$ has a

unique marginal $-1/2$ except $\mathrm{F}_5+\mathrm{F}_5$; the same half argument excludes those cases. In $\mathrm{F}_5+\mathrm{F}_5$, the marginals are a or $a/2$ and -a or $-a/2$. Their $\sum -1/2$ forces either $a=1$ with M marginal $1/2$, or $a=-1$ with M marginal -1. The matching H^0 is then at most 0 or $3/2$ respectively, while (E) demands 1 or 3. This exhausts O15.

4. Complete retained 28 *budgets.* Every Z at u can change only its own group array. At most one of its blocks is in M. The two other M arrays remain nonzero; each outside Z also creates its own nonzero outside array. Consequently

$$A \geq z+1$$

For $z \geq 3$, each class needs two N and an actual star is necessary. The five inactive centers contribute 25 to sum f, giving $\Pi \geq 14+3z$. Every Z block costs a loop, so

$$R \geq 14+3z+(z+1)+z = 15+5z \geq 30$$

Only $z=1$ or 2 remain. For $z=1, \Pi \geq 18, C \geq 6$ and $A \geq 2$. $A=2$ forces the matching array to cancel, so $Z=2$ and the proved value 2 bound gives $\Pi \geq 19$. The complete $R=28$ tuples are

Π	A	C	X
18	3	6	1
18	3	7	0
18	4	6	0
19	2	6	1
19	2	7	0
19	3	6	0
20	2	6	0

Outside $M, A \geq 4$, so only $\Pi=18, A=4, C=6$ occurs. $A \geq 3$ does not by itself assert whether the M array cancels; every allowed actual support is kept in this table.

For $z=2$, the proved two-class $f=13$ bound gives $\Pi \geq 21$. $A \geq 3$ and $C \geq 2$ leave precisely these ten tuples:

Π	A	C	X
21	3	2	2
21	3	3	1
21	3	4	0
21	4	2	1

Π	A	C	X
21	4	3	0
21	5	2	0
22	3	2	1
22	3	3	0
22	4	2	0
23	3	2	0

$A = 3$ forces one Z in M with coefficient 2. If neither Z is in $M, A \geq 5$.

5. Complete one-Z exclusion. When $\Pi \leq 19$ and $C = 6$, the active $f \leq 11$. A nonzero separating marginal would require $f \geq 13$, so every nonzero marginal is on a half containing its source mate t. Hence $B_{ut} = -2\Gamma = 1-2Z$. Every inactive $f \leq 8$ and its B row is zero / one. SZ6 gives zero flow at source U. Equation (F) therefore implies

$$(1 - 2Z + B_{tu})/2 = w_U - Z,$$
$$B_{tu} = 2w_U - 1$$

For matching corrections w_U is 2 or 0, requiring $B_{tu} = 3$ or -1. Both contradict inactive zero / one lightness. This excludes all four $C = 6$ tuples with $\Pi \leq 19$, including $X = 1$ and the outside-M helper.

At $\Pi = 19, A = 2, C = 7$, the Z is 2. Its active f is at least 11 and at most 11, and all inactive centers have $f = 5$. Every crossed glue at u is therefore $\pm 1/2$(the proved F_5 magnitude rule at its opposite inactive line). Z13 excludes it. This local contradiction has no flow or mixed premise.

At $\Pi = 18, A = 3, C = 7$, the line sum is 34. Inactive strict excess begins at 2, so the active f is 9 and all other f are 5. The proved $f = 9$ law forces $Z = 1$, one dense F_5 main line with marginal $+1/2$ and glue vector $(-1/2, +1/2, +1/2)$, and one glue-free F_4 line. In that private partition, the positive glues force their opposite $f = 5$ columns to have all three glues nonzero; hence both other row centers are selected there as well. All five inactive marginals are $-1/2$. The three row marginals $\sum +1/2-1/2-1/2 = -1/2$, but the three column marginals $\sum -3/2$. They sum the same glue matrix, contradiction. This treats the new $C = 7$ budget using its explicit unchanged local hypotheses.

Finally $\Pi = 20, A = 2, C = 6$ has $Z = 2$. Every inactive $f \leq 7$. If the active center has no nonzero separating marginal, SZ6 gives the same impossible source equation $B_{tu} = 3$. Otherwise $f \geq 13$; the line sum is 38, so active $f = 13$ and all inactive $f = 5$ are forced. Their crossing glues again force the half-grid, and Z13 excludes it. All seven one-Z tuples are closed.

6. Complete two-Z exclusion.

6.1 Consequences when $C \leq 3$. Only the two Z blocks can flow: an outside Z-free flowing block would cost two loops in addition to their required two. Both c vectors vanish at u. Projection (P) gives $a + b = 0$ and $\Gamma = -1/2$. Equal sources give zero projected right side; their distinct nonparallel c vectors force both flows zero, requiring two loops per Z block and $C \geq 4$. Thus the sources differ whenever $C \leq 3$.

6.2 $\Pi = 21$. The active f is 13 and all inactive f are 5. $A = 3$ implies matching $Z = 2$, contradicting T13, regardless of C or X. This includes the $C = 4$ cell, where flow might otherwise leave the Z blocks.

For $A \geq 4$ the allowed tuples have $C \leq 3$. If either source is the M-edge at u, O15 applies, including when that helper belongs to an outside-M group. Otherwise both sources have correction weight 0. The opposite-class $f = 13$ subcase has common F_4 and singleton $F_4 + F_5$, so its unique marginal is $-1/2$ and every B entry is zero or one. At those two sources, equation (F) requires $H^0 = -a$ and $H^0 = +a$. Their nonnegativity forces $a = 0$, contradiction. Thus every $\Pi = 21$ tuple is closed. Helper-group membership and matching-source membership are distinct throughout this step.

6.3 $\Pi = 22$. Here $A \leq 4$, so one Z is in M; $C \leq 3$. The active $f \leq 15$ and every inactive $f \leq 7$ by subtracting the active minimum 13 and the other four inactive minima 20 from total line cost 40. Their B rows are zero / one. O15 closes all three tuples, including $C = 3$ and $X = 1$; no idle-loop assumption was made.

6.4 $\Pi = 23, A = 3, C = 2$*: the largest private budget.* The matching Z is $a = 2$, the other $b = -2$, and their sources differ. The line sum is 42. The active $f \leq 17$ and every inactive $f \leq 9$, so all inactive B rows are zero / one. If active $f \leq 15$, O15 applies. Active $f = 16$ would leave inactive $\sum 26$, an excess of one over 25, impossible. Thus only $f = 17$ remains, forcing all inactive $f = 5$ and every active glue to $\pm 1/2$.

Let N be the N count at u. $\Gamma = -1/2$ requires a glue and center mass requires a star, so $N \leq 7$. For every case below the active layer has only one nonzero-marginal line, of marginal $-1/2$. Its B row is zero / one, and (E) would require $H^0_{u,t_2} = 2$, though $H^0 \leq 1$.

- **N4:** There are three main N-lines; four would cost 16 and leave only one unit for a glue, producing a lone F_5 marginal of absolute value 1 or 2, not 1/2. With three lines, each nonzero singleton marginal costs at least 7(F_5 is incompatible with the half-grid, F_6 is zero). A nonzero common marginal costs at least 9: if the two N cancel it is an N-free marginal plus their cost 4; otherwise both actual-star positions remain nonzero because their amplitudes are ± 2 and effective stars lie in $[-1, 1]$. Two nonzero singleton lines would cost at least $4 + 7 + 7 = 18$; a common and singleton would cost at least $9 + 7 + 4 = 20$. An extra N-free nonzero line costs

5, and cannot coexist with another nonzero marginal within 17. At most one marginal is nonzero.

- **N5:** Main counts are $3+2$ and require at least four N-lines costing at least 16. At 17 there is room for only one line to have glue; every F_4 is glue-free. No extra N-free line fits. The marginal is unique.
- **N6:** Main $3+3$ or $4+2$ would require five lines costing 20. Hence there are two N per main class and one balanced pair. At cost 17, either there is one glue and two stars (unique marginal), or three glues and one star. In the latter, their signs are two negative and one positive because $\Gamma = -1/2$; the sole actual star equals 1. Effective star entries lie in $[-1/2, 1]$, and aggregate N entries in $[-3/2, 1]$. Neither an isolated $+2$ nor -2 main N can fit. The two main N on the common line must cancel at the same cell, and the balanced pair must cancel or modify the two singleton main N at their cells. Their residual aggregates are k,-k on the singleton lines, and zero elsewhere. A line with aggregate zero and no star has no glue. If the star is on a zero-aggregate line, that line uses all three glues and is the only nonzero marginal. If it is on a singleton, the other star-free singleton either has marginal 0 or needs three glues of form(-r,r,r). It therefore uses all three whenever nonzero. Again there cannot be two nonzero marginals. The exact collapsed-line control has 11 permitted patterns, all with just one nonzero marginal.
- **N7:** Its cost 14 plus a glue and an actual star exhausts 17, so there is exactly one glue and one star; the marginal is unique.

This excludes the last two-Z tuple. No rational coefficient of a three-N class or balanced pair was assumed to be an integer.

S15. Further matching cases at retained support twenty-nine

The following derivations expand the path, disjoint-source, high-attachment, and nonunit-flow cases used in S9–S10. They retain the condition $R \le 29$ throughout. The source-array contradictions and the sixty-eight affine plus sixteen cost obstructions in S10.4 are used in their respective scopes.

S15.1. **Initial reductions at retained support twenty-nine.**

Scope. Matching $R \le 29$ in the stated support-minimum representative.

Obstruction. Exclude a sole Z at two or three attachments, excess two-center helpers, a confined two-Z block, and six attachments.

Conclusion. The residual families pass to S8–S10 and the following components.

Throughout this subsection, the correction graph is a matching M, $R \le 29$, and the representative minimizes (R, z, d) at fixed correction data, as in S4. In particular $C \ge z$. The local line costs and class laws are those of S5. Balanced N, signed coefficients, mixed columns and idle loops remain allowed unless a displayed budget excludes them.

A sole physical helper with two attachments.

Lemma SL9. *There is no such representative with $m = 2$ and $z = 1$.*

Proof. A sole Z costs $C \geq 6$. First let $d = 0$. Its center costs at least eight, the helper-free active center at least nine, and the four inactive centers at least five each. Hence $\Pi \geq 20$. A Z outside M leaves $A \geq 4$, giving $R \geq 30$. A noncanceling M-Z leaves only $(\Pi, A, C, X) = (20, 3, 6, 0)$; its center has $f \leq 9$, the other active center $f \leq 10$, and every inactive center has $f = 5$. A canceling M-Z has coefficient $a = 2$, so its center costs at least eleven. The only budget is $(21, 2, 6, 0)$, with costs $(11, 9; 5, 5, 5, 5)$.

In both cases the helper center u has no nonzero separator for its source $\{u, p\}$. The sole-helper source-flow lemma in Section S14.8 applies at exactly $C = 6$, so the flow divergence at this source is zero. Equations (S4.6) and $B_{up} = 1 - 2a$ give

$$H^0_{up} = 2 - a, \qquad B_{pu} = 2H^0_{up} - B_{up} = 3.$$

An inactive p has $B_{pu} \leq 1$; a helper-free active p of cost at most ten has $B_{pu} \leq 2$. Both contradict the displayed equality.

For $d = 1$, distinct D and Z blocks give a multiclass endpoint and

$$\Pi \geq \left\lceil \frac{13 + 8 + 20}{2} \right\rceil + 2 = 23.$$

Since $A \geq 1$ and $C \geq 6$, this is too expensive. In a common block, two cost-eight endpoints would have equal class values $1/2$ and hence $Z = 0$. Thus $\Pi \geq 21$, leaving $(21, 2, 6, 0)$. The common block must be in M with $\beta = 2$. The endpoint cost sum is at most eighteen. The only possibilities are $(8, 10)$ and $(9, 9)$: the cost-at-most-ten class list rules out class sum two in the first, while the second gives class values $(1, 1)$ and again $Z = 0$. For $d \geq 2$, both endpoints are multiclass and $\Pi \geq 23 + d + z$; adding $C \geq 6$ already exceeds 29. □

A sole physical helper with three attachments.

Lemma SL10. *There is no such representative with $m = 3$ and $z = 1$.*

Proof. The initial integer envelope leaves only

$$(m, d, z, \Pi, A, C, X) \in \{(3, 0, 1, 21, 2, 6, 0),\ (3, 1, 1, 22, 1, 6, 0),\ (3, 2, 1, 23, 0, 6, 0)\}.$$

For $d = 0$, the active costs are at least $8, 9, 9$, and the inactive cost is at least fifteen. Including the helper gives $\Pi \geq 22$. For $d = 1$, $A = 1$ requires two canceled matching arrays. The sole D must have value one in one matching block and the sole Z value two in another. Their three distinct active endpoints cost at least $9 + 9 + 11$, giving $\Pi \geq 24$. For $d = 2$, $A = 0$ requires one helper in each of three matching blocks, namely two unit D and one $Z = 2$. Their sources require five distinct active endpoints, contradicting $m = 3$. □

Too many helpers on two attachments.

Lemma SL11. *If $m = 2$ and Z is present, then the only remaining helper counts are $(d, z) = (0, 2), (0, 3), (1, 2)$.*

Proof. Lemma SL9 excludes $z = 1$. If $d \geq 2$, the two endpoints have at least two nonzero classes each, so $\Pi \geq 23 + h$. Now $h = d + z \geq 4$, $A \geq 1$ and $C \geq 2$, giving $R \geq 30$. Thus $d \leq 1$.

Suppose $h \geq 4$. If both endpoints have only one nonzero class and $d = 0$, each endpoint has at most one Z, so $h \leq 2$. If $d = 1$, that D fixes the class at both endpoints, and the only permitted helpers are D, Z_u, Z_v in its block, so $h \leq 3$. Consequently some endpoint is multiclass and $\Pi \geq 21 + h$. If $A = 1$, the two matching edges touched by the active centers must both cancel. The centers therefore lie on different matching edges, each carrying a sole $Z = 2$. Every additional helper supplies another class at one of these centers. Such a center cannot have $f = 13$: the two-class cost-thirteen law excludes a value-two class, and three classes require at least six N and an actual star, costing fourteen. The other center costs at least eleven. Hence $\Pi \geq 23 + h \geq 27$ and $R \geq 27 + 1 + 2 = 30$. Thus $A \geq 2$. Since $z \geq h - 1$, the normal form now gives

$$R \geq (21 + h) + 2 + (h - 1) = 22 + 2h \geq 30.$$

Therefore $h \leq 3$, proving the stated list. □

Two physical helpers in one block.

Lemma SL12. *For $m = 2$ or 3, exactly two global Z columns cannot occupy one block.*

Proof. Their source endpoints lie in K. Every D source also lies in K, which contains at most one complete matching edge. Thus at most one matching array can cancel and $A \geq 2$. If $C \leq 5$, the one-block flow lemma in Section S13.5 forces equal endpoint Z values, contrary to the lexicographic normal form. For $m = 3$, $\Pi \geq 20 + h \geq 22$, so $C \leq 5$ already.

Let $m = 2, d = 0$. Then $\Pi \geq 20$. A $C = 7$ boundary would force $\Pi = 20, A = 2$, but the two cost-eight endpoints give $Z_u = Z_v = 1/2$ and $\beta = 1$, which cannot cancel a matching array. At $C = 6$, only $(\Pi, A) = (20, 3)$ or $(21, 2)$ remain. The first again forces equal half-values. In the second $\beta = 2$ and endpoint cost at most eighteen leave $(9, 9)$ with equal values one; the alternative $(8, 10)$ cannot have class sum two. Both violate the normal form. For $d = 1$, a different D block gives two classes at both endpoints, $\Pi \geq 26$, and $A + C \geq 4$. In the common block $\Pi \geq 21$; the sole $C = 6$ boundary has $\Pi = 21, A = 2$ and two cost-eight endpoints, whose class sum is one. It cannot cancel a matching array. The remaining budgets have $C \leq 5$. Higher d was excluded above. □

Six attachments.

Lemma SL13. *No matching identity of retained support at most 29 has six attachments.*

Proof. Always $\Pi \geq 27$. With $Z = 0$, repeated D incidence gives $\Pi \geq 30$. For disjoint D sources, if t coefficients differ from $1/2$, the endpoint costs give $\Pi \geq 27 + t$. At $\Pi = 27$ all three matching arrays are nonzero, so $A \geq 3$. At $\Pi = 28$, the budget gives $A \leq 1$ and $C = 0$; two array cancellations need two unit D, hence $t \geq 2$ and $\Pi \geq 29$. The case $A = 0$ requires three unit matching D and $\Pi \geq 30$. These arguments also allow helper-free balanced attachments and mixed slack.

With Z present, $\Pi \geq 27$ and $C \geq 2$ force $(\Pi, A, C, X) = (27, 0, 2, 0)$. The helper charge gives $h \leq 3$, whereas canceling all three matching arrays requires $h \geq 3$. There is exactly one helper in each matching block: every D has value one and every Z value two. Each active center has nonzero Γ, including helper-free balanced centers, and therefore $f \geq 9$. Consequently $\Pi = \frac{1}{2} \sum f + h \geq 27 + 3 = 30$, a contradiction. □

These lemmas exclude the new retained-29 boundaries explicitly. They do not extrapolate a retained-28 contradiction of the form $R \geq 29$. The remaining families and the precise integer filters are listed in S8.

S15.2. **Two intersecting doubled-helper sources.**
Scope. Matching $R \leq 29$, $Z = 0$, two intersecting D sources on three active centers.
Obstruction. The shared-endpoint cost forces $C = 0$. Source balance leaves a unit case whose shared-center marginals disagree.
Conclusion. No two-D path.

Let the shared center be s and the leaves v,w. The sources are $U =$ sv, $V =$ sw, with coefficients a,b. At most one D block belongs to M. The shared center costs at least 13, the leaves at least 8 each, and the three inactive centers at least 15, so $\Pi \geq 24$. All possible budgets have $C \leq 1$; $Z = 0$ excludes $C = 1$, hence $C = 0$. Mixed terms vanish in the source equations. When the U block belongs to M, the intersecting V source cannot contribute as a side on U, and F gives

$$H^0_{sv} = 2 - 2a$$

If the v leaf has no nonzero separator, its directed entry is $1 - 2a$, so the required shared directed entry is

$$B_{sv} = 3 - 2a. (P)$$

We use the P14 lemma only for positive class pairs $(1/2, 1/2), (1, 1/2), (1, 1)$ and shared cost at most 14. It gives $B \leq 0$. We never extend that lemma to cost 15 or 16.

1. Main-line geometry and signed small-line facts. Two distinct source-prefix class line sets share exactly one private line, on half $\{s, v, w\}$. Each minimal class has one N on each of two distinct main lines. Four minimal N therefore occupy either three or four lines. In the three-line case there is one shared line, one U-only line containing v but not w, and one V-only line containing w but not v. Five main N occupy at least four lines. Additional balanced classes require two N. All these statements are checked from the literal metadata for 540 class pairs and 8640 minimal main-pair placements.

At shared cost $13 - -16$, if there are exactly four N and $\Gamma \neq 0$, four occupied lines cannot fit at cost ≤ 15; at cost 16 they would all be F_4 with zero marginal and hence cannot fit either. An additional N-free nonempty line costs at least 5, beyond the three main lines' minimum 12 at these costs. Thus the three-line description applies.

For later signed cases, a shared line carries N values $(-1, +1/2)$ or $(-1, +1)$, possibly at the same V position. The exact local affine check over all three unrestricted glue variables shows:

- With $(-1, +1/2)$, cost at most 6 permits only an F_6 line with no glue and marginal zero.
- With $(-1, +1)$, cost at most 6 permits only F_4 or F_6, again with marginal zero. There is no F_5 form.

These facts follow directly as well. At F_4 the aggregate N must vanish. At F_5, no star is available and one glue would have marginal $-\sum(N)/2$; the required three entries do not have one-glue support. At F_6, either there are no glues and one star, or two glues and no star. The two-glue case with $N = (-1, +1/2)$ would require marginal $1/4$ but three nonzero glues. With $N = (-1, +1)$ it has marginal zero. The checker covers 40 masks and preserves all three faithful forms.

A one $-N\mathrm{F}_5$ with $N = -1$ has marginal 1 or $1/2$. With $N = +1$ it has marginal -1 or $-1/2$. F_4 and one $-N\mathrm{F}_6$ have marginal zero.

2. $\Pi = 24$, including both D groups outside M. Both leaves are $f = 8$ with $a = b = 1/2$, the shared center $f = 13$, and every inactive $f = 5$. P14 makes its B entries nonpositive. If either source is an M-edge, F requires $H^0 = 1$ there whereas its $f = 8$ leaf has $B = 0$ and the shared entry is nonpositive. If neither source is in M, the shared center's M-mate is inactive. That M-edge is neither a source nor a side of either D block, and $H^0 \leq 1/2$, contrary to weight 2.

Thus $\Pi = 24$ is impossible regardless of group placement or mixed slack. Two outside-M blocks require $A \geq 5$ and therefore allow only $\Pi = 24$; that whole branch is excluded. Hence one block, called U, lies in M.

3. A nonunit M-D: the $\Pi = 25$ boundary. $A \geq 4$, so $\Pi \leq 25$. At $\Pi = 25$ the doubled line cost is 46. Both leaves have cost at most 10. The low-cost class list and the $f = 13$ equal-absolute-value law leave the following cases.

Positive values with shared cost at most 14 are handled by P14: the M leaf is a half class of cost at most 10, with $B = 0$, whereas (P) asks for $B_{sv} = 2$. If a leaf is negative, its cost is at least 10, the other leaf is $f = 8$ and the shared center is $f = 13$. Equal absolute values force the pair $(1/2, -1/2)$, in either order. Its shared Γ is $-1/2$. Four N on three lines at cost 13 give costs $4, 4, 5$; the sole nonzero marginal is $-1/2$. Thus its B row has entries 0 or 1. Equation (P) requires 2 or 4, impossible. Negative value -1 would violate the equal-magnitude law.

The only new positive case is $a = b = 1/2$, shared cost 15, both leaves $f = 8$, and all inactives $f = 5$. Every shared glue has magnitude $1/2$ because the leaves are glue-free and every inactive F_5 crossing glue has that magnitude. Shared Γ is $1/2$, and(P) requires $B_{sv} = 2$.

Six N leave one glue and one star at cost 15. That glue is $+1/2$, so no positive B entry is possible. Five N would occupy four lines and cost at least 16. With four N there are precisely the three main lines of item 1 and

$$g + 2S = 7, \text{positive}_{\text{ glues}} - \text{negative}_{\text{ glues}} = 1$$

With $g = 1$ or $g = 3$, total negative glue mass is less than 1, insufficient for $B_{sv} = 2$. With $g = 5$,S1, there are three positive and two negative glues. The together-U marginal must be -1, so both negative glues and no positive glue are on together-U lines. All three positive glues must occupy the unique V-only line. Its N is $-1/2$ and its three glues are $+1/2$, so its actual stars are $(-1/2, -1, -1)$, up to permutation. Three nonzero stars exceed the total S1 budget. This closes $\Pi = 25$.

4. A unit M-D: reduce $\Pi = 26$ to four coefficient cases. For $a = 1, \Pi \geq 25$ and $A \geq 3$ leave $\Pi = 25$ or 26. At $\Pi = 25$ the two possible positive patterns are shared / leaf costs $(14, 9, 8), b = 1/2$, and $(13, 9, 9), b = 1$. P14 excludes both; negative values do not fit.

At $\Pi = 26$ the doubled line cost is 48. The unit leaf v has cost at most 12. Lemma SL5 (J12) confines its glues to together-U lines, so its $B_{vs} = -1$. Equation (P) therefore requires

$$B_{sv} = 1.(U)$$

The other leaf has cost at most 11. If it costs 11, shared cost 13 and the equal-absolute-value law force $b = \pm 1$. At costs at most 10 the list gives b in $\{1/2, 1, -1/2, -1\}$. Thus these four values cover arbitrary rational b. Cases $b > 1$ or other unlisted values are not discarded by a sign assumption; their necessary $f = 11$ boundary fails the equal-magnitude law.

$b = -1/2$. The shared cost must be 14: cost 13 would have unequal class magnitudes. Both leaves and all inactives use costs $9, 10, 5, 5, 5$. Shared Γ is 0. Six N at cost 14 leave no glue and give $B = 0$. Five main N exceed 14. With four N there are three lines. The shared two-N line has N values

$(-1, +1/2)$, so item 1 permits only its glue-free F_6 form; the two other lines must be F_4. All marginals vanish, again giving $B_{sv} = 0$, contrary to(U).

$b = -1$. Shared cost is 13 or 14. Six N would leave no glue, incompatible with $\Gamma = -1/2$; five N exceed the budget. With four N, the shared two-N line cannot be F_5 and has marginal zero. If it is F_6, both other lines are F_4, contradicting $\Gamma = -1/2$. Thus it is F_4.

At shared cost 13, the only nonzero marginal is the V-only F_5 value $-1/2$, so $B_{sv} = 0$. At cost 14 both sole-class lines must be F_5, with marginals r_U in $\{1, 1/2\}, r_V$ in $\{-1, -1/2\}$. Their $\sum -1/2$ forces $(r_U, r_V) = (1/2, -1)$, hence $B_{sv} = -1$. Neither equals the required 1. This includes the other leaf's possible $f = 11$ cost.

$b = 1$. P14 handles shared costs at most 14. The only larger case is shared 15, both leaves $f = 9$ and all inactive $f = 5$. All shared glues have magnitude $1/2$, and $\Gamma = 3/2$. Six N leave room for only one such glue, insufficient for Γ. Five N exceed 15. For four $N, g + 2S = 7$. With $g = 3$ all glues are positive and $B_{sv} \leq 0$. With $g = 5$ there are four positive and one negative glue. To make $B_{sv} = 1$, the negative glue and no positive glue must lie on together-U lines. Four positive glues would have to occupy the single V-only line, which has only three positions. Contradiction.

$b = 1/2$. P14 again handles shared costs at most 14. Shared 15 cannot complete the even total 48: unit $f = 10$, half-class $f = 9$ and inactive $f = 6$ do not exist. The only remaining case is shared 16, unit leaf $f = 9$, half leaf $f = 8$, and inactive costs $5, 5, 5$. All shared glues have magnitude $1/2$ and $\Gamma = 1$.

Six N allow at most two glues and one star; $\Gamma = 1$ forces both glues positive, giving $B_{sv} \leq 0$. Five main N would occupy four F_4 lines at cost 16, with $\Gamma = 0$. Four N give the three main lines and $g + 2S = 8$. With $g = 2$ both glues are positive. With $g = 4$,S2 there are three positive and one negative glue. Condition(U) places the negative glue and no positive glue on together-U lines; the three positive glues fill the V-only line and require three actual stars, more than S2.

With $g = 6$,S1 there are four positive and two negative glues. To have together-U marginal $-1/2$, either one negative and no positive glue lie there, leaving five glues for the V-only line, or two negative and one positive glue lie there. The latter leaves three positive glues on the V-only line, again requiring three stars rather than S1. Both alternatives are impossible.

This exhausts the unit case and every two-D path at matching $R \leq 29$.

S15.3. **Two disjoint doubled-helper sources.**
Scope. Matching $R \leq 29$, $Z = 0$, two disjoint D sources on four active centers.
Obstruction. Private and array costs force zero flow. Source equations exclude nonunit values; the two unit helpers demand more actual-star mass than is available.
Conclusion. No disjoint two-D case.

The initial $R = 29$ map reduces this branch to two disjoint D sources U,V, with coefficients a,b. One D is excluded by E23, or U25 when its M array cancels; repeated / intersecting sources and $d \geq 3$ are excluded by their actual $R = 29$ degree / array costs. $\Pi \geq 23, A \geq 1$, and nonzero flow would cost $C \geq 6$, exceeding 29. Thus coordinate flow is zero.

Write q,r for the two inactive vertices. The exact source equations are

$$w_U = 1 + \mathrm{NT}_U + b \cdot [U \mathrm{is} a \mathrm{side\ of the\ } V \mathrm{block\ }],$$
$$w_V = 1 + \mathrm{NT}_V + a \cdot [V \mathrm{is} a \mathrm{side\ of the\ } U \mathrm{block\ }]. \quad \text{(T2)}$$

Here NT is the sum of separating marginals at the two source endpoints. All mixed terms vanish in these equations. A nonzero separator costs at least 13. A unit-class cost 13 separator has marginal $1/2$, with all other marginal lines at that endpoint zero: its two main lines cost 4 each and its separator costs 5. This statement allows balanced collisions.

1. Local incidence and glue bounds. By Lemma SL5 (J12), the stars at a sole unit-class center of cost $f \leq 12$ lie on its two together lines. Their source pairs contain that center's class mate; here they can therefore be neither U nor V. The half-class version applies at $f \leq 10$.

G11. All glues at a unit-class center of cost at most 11 have magnitude at most 1, and belong to $\{0, \pm 1/2, \pm 1\}$.

At $f = 9$ they have magnitude $1/2$; $f = 10$ is impossible. At $f = 11$, four N force one glue, equal to $\Gamma = 1/2$. Three N cost at least 12. For two main $N = -1$ on the two lines, the exact six-variable affine check covers all 456 faithful-support masks of cost 11. Its eight surviving forms have precisely the stated glue-value set. Their equality equations and coefficients are saved; this is an unrestricted rational local check, not coefficient sampling.

Every star in block(M,U) is centered outside U, and each outside center has exactly one star column in that block. At an inactive center, such a star lies on half U plus that center. These literal facts are checked for all 45 blocks.

Two inactive F_5 centers together contribute at most 1 to a fixed block (M,U) whose source is disjoint from them. Each individual contribution is $0, 1/2$ or 1. A contribution 1 means a dense F_5 line on half $U+q$. Its glue to r forces r's unique nonempty line onto the complementary half, whose source is V, so r contributes zero to(M,U). Otherwise both contributions are at most $1/2$. This proves the bound without constraining the other centers' coefficients.

2. $\Pi \leq 25$*: all sixteen forced equality forms are excluded.* The doubled line budget is at most 46. No endpoint can reach separating cost 13 while the other three cost at least 8 and the inactives cost 10. Thus $\mathrm{NT}_U = \mathrm{NT}_V = 0$. T2 forces both side incidences, hence both D belong to the unique matching containing U,V. It also forces a,b in $\{-1, 1\}$. A negative class costs at least

10, so $\Pi \le 25$ forces $a = b = 1$, their group $M, \Pi = 25$, four active $f = 9$ centers and two inactive $f = 5$ centers.

The four active glue lines are dense and in a common split with U and V on opposite sides. A third vertex from the other D pair would force a separating glue at its opposite $f = 9$ mate, which is impossible. All four dense lines are then forced by their crossing glues. Relabel the split as $014|235, U = 01, V = 23$. The literal main-N positions force

$$L = (1/2) \cdot [[1, 1, -1], [1, 1, -1], [-1, -1, 1]]$$

At each active center, the dense-line N is fixed. Its other h side has exactly two possible N, one of which must be chosen. No other private support fits. These give exactly $2^4 = 16$ equality forms. The other split and all labels are equivalent under permutations of the inactive pair and the matched pairs. This covers all relative N choices.

The checker derives these 16 cores from literal incidences. Twelve have five nonzero B demands, two have six, and two have four. All require $A \ge 4$; $R = 29$ thus forces $A = 4, C = X = 0$. For each four-demand core every A must lie in one of those four blocks, giving 16 allowed columns. All 16 are allowed simultaneously in the exact affine equations. Rational row duals prove rank 16 and force a unique solution with six nonzero A. Each dual is stored with its original literal / array row labels and exact weights and is checked by multiplication. Thus none of the 16 forms admits $A = 4$. No fixed-core conclusion is extrapolated to other cores; all equality forms have been covered explicitly.

3. $\Pi = 26$. $A \le 3$ forces at least one D block into M. If one is outside, the inside D must be a unit, since otherwise four arrays remain nonzero.

If both coefficients are nonhalf, all endpoints cost at least 9. A separator would require line cost at least $13 + 27 + 10 = 50$, beyond 48. Thus both NT vanish. T2 forces a common matching group, and the inside block makes that group M; consequently $a = b = 1$.

If both coefficients are halves, both blocks must be in M since no M-array cancels. T2 requires $\mathrm{NT}_U = \mathrm{NT}_V = 1/2$, hence one separator at each pair. Even the weaker cost $13 + 8$ at each pair gives total $52 > 48$.

Suppose only b is a half. Its source cannot have a separator: the required total would be at least $13 + 8 + 18 + 10 = 49$. If both blocks are in M, T2 forces $a = 1$. If the half block is outside M, the other is already a unit in M. A half source outside the M matching would give $w_V = 1$ from T2, impossible; if V is an M-edge, its outside group cannot also contain U. Then $\mathrm{NT}_U = 1$ is required, whereas a unit endpoint at this budget has cost at most 13 and NT_U is 0 or $1/2$. This excludes the outside half block.

In the remaining in-M unit / half case, $\mathrm{NT}_U = 1/2$ forces the exact costs $13, 9$ on $U, 8, 8$ on V and $5, 5$ inactive. The cost 13 separator has marginal $1/2$ and is F_5. A dense separator would cross the $f = 9U$-mate on a separating

line, impossible. Its single positive glue must therefore meet an inactive F_5 line. That line must be dense. It forces every center opposite it to carry glue, including a glue-free V endpoint: the opposite half contains the separating U endpoint but excludes its mate, and cannot exclude both V vertices as well. Contradiction.

It remains $a = b = 1$ in M. At $\Pi = 26$ every endpoint costs at most 11, so Lemma SL5 excludes its stars from both D blocks. An inactive $f = 7$ would have the other five centers at their minimum costs; all its crossing glues would have magnitude $1/2$, making its supposed F_7 line F_5. Hence both inactive centers are $f = 5$, and each D block has actual star sum at most 1. Both therefore require strictly positive alternating sums. With $A \leq 3$ the nonzero remaining M-array consumes one direction; outside-M directions can only be an opposite-coefficient pair in one zero-total array. That pair cannot supply positive sums to two different blocks, and group-M alternatings belong to neither D block. This excludes $\Pi = 26$, including its possible idle-loop or mixed slack.

4. $\Pi = 27$ *with a nonunit second coefficient.* $A \leq 2$ forces both D in M and at least one unit, say $a = 1$. If b is nonhalf, each U endpoint has cost at most 13. There is at most one nonzero separating line at U, so NT_U is 0 or $1/2$. T2 then forces $b = 1$ or $1/2$. Thus a genuinely nonunit b must be $1/2$.

For $b = 1/2, \mathrm{NT}_U = 1/2$ requires a unit endpoint of cost at least 13. The V-endpoint cost sum is at most 18, hence one has $f = 8$ and is glue-free, and the other has cost at most 10. Their stars stay on together lines for V and contribute nothing to block(M,U). Neither U endpoint can have source U. All A are in M, so the U block requires actual star $\sum 2$ solely from the inactive centers.

Each inactive costs 5 or 7. A star with source U needs half U plus that inactive vertex. Its opposite half contains the glue-free V endpoint. Thus neither a dense F_5 nor an F_7 line can occupy that half. Only a sparse F_5 remains, contributing at most $1/2$ to the particular block. The two inactives supply at most 1, contradicting the required 2.

The remaining $\Pi = 27$ case and every $\Pi = 28$ case have two unit D in M.

5. Two unit D *in* $M, \Pi \leq 28, A \leq 2$. All A are in group M: its remaining nonzero array needs one direction, while an outside zero-total array would need two more. Thus both D blocks require actual star $\sum 2$, for combined $\sum 4$. Loops and mixed columns vanish in these B aggregates.

The minimum doubled private line cost is 46, with four unit $f = 9$ centers and two inactive $f = 5$ centers. At $\Pi \leq 28$ the excess is an even integer at most 6. Unit $f = 10$ and inactive $f = 6$ do not occur.

If a unit center has cost at least 13, it is the only such center. It has cost 13 or 15; cost 14 would leave an impossible one-unit excess. All other unit centers have cost at most 11 and contribute no stars to either D block. An

inactive $f = 7$ could then have only one partner with non-half-magnitude glues. But its N-free F_7 form $(r, t, -t), r = -1/2$, needs two such partners because t is neither 0 nor $\pm 1/2$. Therefore both inactives are $f = 5$. In the D block whose source contains the high-cost unit center, neither endpoint contributes, the other unit centers contribute zero, and the two inactives together contribute at most 1 by item 1. The required $\sum 2$ is impossible.

Otherwise all unit centers cost at most 12 and Lemma SL5 excludes all their stars from both D blocks. We bound the combined contribution of the two inactive centers, preserving possible negative stars and nonempty zero-marginal lines. The even cost excess gives the following exhaustive classification, also checked as 99 small integer cost profiles.

- If an inactive has $f = 11$, every other center is at minimum cost. All its glues therefore have magnitude $1/2$. Its only possible line costs are one nonzero-marginal F_5 plus one zero-marginal F_6 : a single N-free line costs at most 9; two nonzero lines cost 10 and cannot gain just one unit. Its total positive star mass is at most $1 + 1/2$. The other inactive contributes at most 1, giving at most $5/2 < 4$.
- An inactive $f = 9$ has one F_9 line. If the other inactive is $f = 7$, all unit centers are $f = 9$, making that F_7 impossible by the two-partner argument. Otherwise the other inactive is $f = 5$ and at most one unit center has $f = 11$. Two of the F_9 line's three nonzero glues have magnitude $1/2$, and the remaining glue lies in $\{\pm 1/2, \pm 1\}$ by G11. Sum $-1/2$ forces that remaining glue also to have magnitude $1/2$, making the line F_5, not F_9.
- An inactive $f = 8$ is accompanied by exactly one other excess 3, either a unit $f = 12$ or another inactive $f = 8$. Its F_8 line has two nonzero glues, at most one with an exceptional partner. The normal glue is $\pm 1/2$; $\sum -1/2$ forces the pair $(+1/2, -1)$. Its stars are $(1, -1/2, 1/2)$, up to permutation, so its contribution to the particular source block is at most 1. Each inactive contributes at most 1, totaling 2.
- Otherwise both inactives have cost 5 or 7. When an $f = 7$ occurs all unit centers have cost at most 11, since an excess 3 together with excess 2 cannot complete the even budget at most 6. Among the two glues t,-t of an F_7 line, at most one meets the other inactive; the other meets a unit center. G11 gives $|t| \leq 1$. Its largest actual star is at most $3/2$. Its single line has one source pair and at most one star in the corresponding M-block. Thus the two inactive contributions total at most 3, below 4.

Inactive $f = 10$ cannot occur: its excess 5 would need an unavailable excess 1. These cases exhaust the budget and contradict the combined star demand. This completes every two-unit case and hence every Z-free, helper-present $m = 4$ case at $R \leq 29$.

S15.4. **Four centers with physical helpers.**
Scope. Matching $R \le 29$, four active centers, $Z > 0$.
Obstruction. Cancellation leaves two or three helpers. Budget equality fixes the endpoint classes, then projection, actual-star supply and source balance contradict them.
Conclusion. All placements are excluded, including paired helpers in one block.

Assume matching corrections, $R \le 29, m = 4, Z$ present, and a support-minimum representative for the fixed correction vector.

Use $\Pi \ge 23, \Pi \ge 21 + h, h = d + z, 2(\Pi - h) = \sum f$, with two inactive centers of cost at least 5. Every occupied Z block needs a loop, normalized $C \ge 2$, and a sole Z needs $C \ge 6$. Actual star mass is 1 at every center.

An M-array cancellation has endpoint class $\sum \beta = 2$. If both endpoints are active, their joint cost is at least 18, with equality only for class values $1, 1$ and costs $9, 9$. If just one endpoint is active, its class equals 2 and its cost is at least 11. These local cancellation bounds permit balanced N and other classes. At cost 8 there is exactly one nonzero class $1/2$; a second class costs at least 13.

1. Remove a sole $Z, A = 0$, and $h \ge 4$. With $z = 1, C \ge 6$ and HA at $m = 4$ gives

$$R \ge 15 + 6 + 3 + 6 = 30$$

Hence $z \ge 2$. If $A = 0$, all three M-arrays cancel. Four active vertices either form two full M-edges, leaving an uncancelable inactive edge, or form one full edge and two crossing edges. In the latter case the full edge costs at least 18, the crossing endpoints at least 11 each, and the two inactive vertices at least 10 together. At least three helpers are needed, so $\Pi \ge 25 + h \ge 28$. With $C \ge 2$ this exceeds 29. Thus $A \ge 1, \Pi \le 26$ and $h \le 5$.

If $h = 5, \Pi = 26$ is forced and the doubled line budget is 42, exactly four active $f = 8$ centers and two inactive $f = 5$ centers. Every endpoint class is 0 or $1/2$, so no M-array can cancel. $A \ge 3$ then exceeds 29.

For $h = 4, \Pi \ge 25$. If $A \ge 2, R = 29$ forces $\Pi = 25, A = 2, C = 2, X = 0$. Again all active centers are $f = 8$ and no array cancels, contradicting $A = 2$. If $A = 1$, at least two M-arrays cancel. Their disjoint sources require active cost at least 36: two full active edges each cost 18; a crossing cancellation instead increases the bound. Including the inactive cost 10 gives $\Pi \ge 23 + h = 27$, above 26. Hence only $h = 2$ or $h = 3$ remain.

2. $A = 1$ at $h = 2$ or $h = 3$. At $h = 2$ there are two Z. To cancel two distinct M-arrays, each must be a sole Z of coefficient 2 at a different active endpoint. Those endpoints cost at least 11 each and the other two active centers, which are helper-free, at least 9 each. Thus $\Pi \ge 25 + 2 = 27$, too large.

At $h = 3$, two canceled M-arrays give $\Pi \ge 26$ by the preceding active-cost bound. $R = 29$ then forces $\Pi = 26, A = 1, C = 2, X = 0$. Equality requires

two fully active M-edges, with four unit classes at four $f = 9$ centers and two inactive $f = 5$ centers. Three Z cannot cover four distinct class centers. With one D and two Z, the only way to cover all four centers is a sole unit D on one edge and the unit endpoint Z pair on the other.

All A belong to M and hence to neither of these two helper blocks. Each block has $\beta = 2$ and requires actual star $\sum 2$. The four active $f = 9$ stars lie on together lines for their own source and cannot have either of the two disjoint source pairs. Only the two inactive $f = 5$ centers can contribute; their nonnegative stars have total mass 2, less than the required 4. Mixed terms and loops vanish in these block aggregates. This excludes $A = 1$. Therefore $A \geq 2$ and $\Pi \leq 25$.

3. $h = 2$: two Z, no D. Two Z at the same center cost at least 13 there, while the other three active centers cost at least 27; with inactive cost 10 and $h = 2, \Pi \geq 27$. Thus the attachments are distinct. The two helper-free active centers cost at least 9 each, giving $\Pi \geq 24$.

At $\Pi = 24$ the two Z centers are $f = 8$, with values $1/2$. No array cancels, so $R = 29$ forces $A = 3, C = 2, X = 0$. Any outside-M Z block would add a nonzero array, so both Z are in M. If they occupy distinct blocks, the loop vectors are constant on each M-edge but the lone-endpoint Z vectors are antisymmetric, contradicting projection. If they occupy one block, the one-block lemma for $C \leq 5$ gives equal Z and zero flow. Source balance then requires $\mathrm{NT} = w_U - 1 = 1$, while their $f = 8$ endpoints give $\mathrm{NT} = 0$. Both arrangements fail.

At $\Pi = 25$ the budget is $A = 2, C = 2, X = 0$, requiring an M-array cancellation. A sole Z of coefficient 2 would cost at least 11 at its endpoint, the other Z at least 8, and the two helper-free centers at least 18 together. With inactive 10 and $h = 2$ this gives $\Pi \geq 26$. The only possible cancellation is therefore a unit Z pair in one M-block, with both endpoints $f = 9$, other active centers $f = 9$ and inactive centers $f = 5$. The one-block lemma again gives zero flow, and the unit endpoints have no separator. Source balance requires 1 instead of 0. This closes $h = 2$.

4. $h = 3$: three Z, or one D and two Z. For three Z, at most three active centers carry nonzero helpers. Hence $\Pi \geq \mathrm{ceil}\,((3 \cdot 8 + 9 + 10)/2) + 3 = 25$. The only remaining budget is $\Pi = 25, A = 2, C = 2, X = 0$. Its active cost is at most 34. A collision of two classes already exceeds this bound, so the Z attach at three distinct centers, with a fourth helper-free center of cost at least 9. There is at most one unit of excess over their minimum costs $8, 8, 8, 9$: at most one class can equal 1, the others are $1/2$, and none can equal 2. No β can reach 2, contradicting the required cancellation.

For one D and two Z, Π is 24 or 25.

At $\Pi = 24$ all four active centers are $f = 8$. The helper incidences must therefore cover four distinct class centers: the two D endpoints and two other Z attachments, each with value $1/2$. No β reaches 2, so $A = 3, C = 2, X = 0$ is forced. All helper blocks must lie in M; an outside block has positive β

and would require a fourth array. Distinct Z blocks are excluded by the same projection argument. For a Z pair on another M-edge, the one-block lemma gives zero flow. The D source has $H^0 = 0$, own $\beta = 1$ and side contribution $1/2$ from the Z block, whereas its required correction weight is 2. Contradiction.

At $\Pi = 25$ the budget is $A = 2, C = 2, X = 0$ and the active cost is at most 34. Canceling an array at a crossing edge would need cost 11 plus the other three active costs $8, 8, 8$, exceeding 34. A full-edge cancellation therefore has the equality costs $9, 9$, with class values $1, 1$; the other two centers are $f = 8$ half classes, and the inactives $f = 5$. All four active centers carry a nonzero class, so the D endpoints and Z attachments are distinct. Every occupied helper block has positive β, forcing all of them into M at $A = 2$.

There are just two arrangements: a unit D plus a half Z pair on another M-edge, or a unit Z pair plus a half D on another M-edge. A mixed $D+$ single-Z cancellation cannot have equal endpoint classes unless the Z is zero. Distinct Z blocks are again excluded by projection. The paired-Z arrangement has zero flow. At the unit-class source, $H^0 = -1$, while own $\beta = 2$ and the other block's side contribution $1/2$ would require $H^0 = -1/2$. This final contradiction closes $h = 3$.

The h and A splits exhaust every normalized Z-present $m = 4$ case at $R \le 29$. The proof uses the minimum choice only for the stated $C = 1$ normalization. It does not discard idle loops merely because flow becomes zero, and deduces any helper or star positivity used above from the stated equality cases.

S15.5. **Five active centers.**
Scope. Matching $R \le 29$, five active centers; $\Pi \ge 25$.
Obstruction. With $Z = 0$, source degrees leave two disjoint D and signed star supply excludes every budget through $\Pi = 28$. With $Z > 0$, incidence and cancellation exceed the budget.
Conclusion. No five-center identity.

Use $R = \Pi + A + C + X, h = d + z, 2(\Pi - h) = \sum f$, and $\Omega = -2e_M, w = 2M$. The baseline is $\Pi \ge 25$ and $\Pi \ge 23 + h$. Actual center star mass is 1. There is one inactive center q, with $f(q) \ge 5$ and no $f = 6$. Helper-free active centers have $f \ge 9$. A nonzero class center has $f \ge 8$; $f = 8$ forces its sole class to $1/2$ and all marginals zero. A sole nonzero class at $f = 9$ is 1, with a glue-free F_4 line and a dense F_5 together line. There is no $f = 10$ for a sole unit class. A nonzero class with a nonzero separating marginal requires $f \ge 13$. These local inputs permit balanced collisions. The literal block equation is star $\sum + A \sum = \beta$.

1. Unit-class stars at cost at most 11 *remain on together lines.* Let a center have exactly one nonzero class, of value 1 and source U, with arbitrary

additional balanced classes, and $f \leq 11$. It has exactly two occupied N-lines, both together lines for U; all its glues and actual stars lie on those two lines.

An actual star is required, so $N \leq 4$. Two main N lie on distinct lines. Three N must all belong to the nonzero class and occupy three distinct lines, each of cost at least 4, exceeding 11. Four N in that class occupy four distinct lines and are also too costly. The other four-N option is a minimal main pair plus one balanced pair. Each N-line costs at least 4, including a two-N collision whose aggregate is zero. Thus at most two such lines fit, and the balanced pair is confined to the two main lines. Any additional N-free nonempty line costs at least 5, beyond the two main lines' combined minimum 8. This proves the assertion.

In particular the actual stars of such a center have source pair equal to its containing half minus that center, hence containing its U-mate. If V is disjoint from U, none can have source U or V. This conclusion uses incidence, not positivity of actual stars at cost 11.

2. $Z = 0$: reduce the source graph to two disjoint D. An M-edge containing q cannot be a D source, so its array remains nonzero and $A \geq 1$. Nonzero coordinate flow would require $C \geq 6$, exceeding $\Pi + A + C \geq 32$. Thus flow is zero, with idle loops still counted.

Repeated or intersecting D sources give $\Pi \geq 28$ by the degree charge. $R = 29$ then forces $\Pi = 28, A = 1, C = X = 0$. But $A = 1$ requires cancellation of two M-arrays and no outside nonzero array. With $Z = 0$, every occupied block has $\beta = 2D \neq 0$, so the D are exactly the two unit columns in those M-blocks. Their sources are disjoint. This contradicts repeated incidence and also excludes $d \geq 3$ on five active vertices.

With one D, E23 gives endpoint cost at least 23. The other three active centers cost at least 27, and q costs 5; hence $\Pi \geq 29$ and $A \geq 2$ exceeds $R = 29$. Therefore exactly two D remain, on disjoint sources U,V, with coefficients a,b. Let w be the remaining helper-free active center. If t of a,b differ from $1/2$, then $\Pi \geq 25 + t$. The possible Π are $25 - -28$.

$\Pi = 25$. Both coefficients are halves; their four endpoints have $f = 8$ and $B = 0$, while w has $f = 9$ and q has $f = 5$. With $A \leq 4$, both D cannot be outside M, since that would require five arrays. At a source U whose block is in $M, H_U^0 = 0$. Its own helper contributes 1 and the other contributes either 0 or $1/2$. Zero-flow F instead requires $H_U^0 \geq 1/2$. Contradiction.

$\Pi = 26$. If both coefficients are halves, $A \leq 3$ forces both D in M. Each endpoint has cost at most 10 and all its marginals vanish. F at either source requires $H^0 = 1/2$ instead of 0.

Otherwise $t = 1$ and equality in $\Pi \geq 25 + t$ forces $f = 9$ at the nonhalf D's endpoints, $f = 8$ at the half D's endpoints, $f = 9$ at w and $f = 5$ at q. The nonhalf coefficient is therefore 1. $A \leq 3$ forces its D block into M: if the unit D were outside, no M-array could cancel, giving $A \geq 4$. At

its source U, the $f = 9$ endpoint rows give $H^0_U = -1$. The other half D contributes either 0 or $1/2$ there, so F requires $H^0_U = 0$ or $-1/2$, impossible. This permits an outside half D and any mixed slack that fits the count.

$\Pi = 27$. $A \leq 2$ forces both D in M and at least one, say a, to equal 1. The total line cost is 50. Each of its endpoints has cost at most 11, since the other costs at least 9, the other two D endpoints at least 16 together, w at least 9 and q at least 5. item 1 puts their total marginal $1/2$ on together lines, giving $H^0_U = -1$. In F the other D contributes b, so $H^0_U = -b$ and $b = 1$.

Equality now forces all four D endpoints to have $f = 9, wf = 9$ and $qf = 5$. There is just one nonzero M-array. With $A \leq 2$, an outside zero-total array cannot carry support, because it would need two columns in addition to the nonzero array. All A are in group M, which has no block-alternating columns in (M,U) or(M,V). Both blocks therefore need actual star mass 2. item 1 excludes the four D endpoints from those blocks. The remaining centers w,q have nonnegative actual stars of total mass 1 each, by the B_9 and inactive $-F_5$ laws. Their combined contribution is at most 2, whereas the required sum is 4. Mixed columns vanish in these block equations; idle loops cannot fit because $C \leq 1$ and $Z = 0$ forbids $C = 1$.

$\Pi = 28$*: the additional direction is covered.* The exact budget is $A = 1, C = X = 0$. Array cancellation again forces $a = b = 1$ in two M-blocks. The minimum doubled line cost is 50; the actual total is 52. Unit endpoints have no $f = 10$, and q has no $f = 6$. Thus precisely one of the following occurs:

- One D endpoint has $f = 11$; the other D endpoints and w have $f = 9, qf = 5$.
- The inactive q has $f = 7$; all five active centers have $f = 9$.
- The helper-free active w has $f = 11$; all D endpoints have $f = 9, qf = 5$.

In case 1, item 1 still excludes all D-endpoint stars from the two D blocks. B_9 and inactive F_5 again supply at most 2 instead of 4.

In case 2, all other centers' nonzero glues have magnitude $1/2$, so every glue at q has that magnitude. Its N-free $f = 7$ line has total marginal $-1/2$ and is its only nonempty line. The F_7 form is(r,t,-t), up to permutation. If every glue has magnitude $1/2$ and $r = -1/2$, the star vector has just one nonzero entry, making the line F_5 rather than F_7. This is impossible. No nonempty zero-marginal line fits in cost 7.

In case $3, w$ has at least two nonzero glues. Here is the full incidence argument. Choose an endpoint u of U and its dense $f = 9$ half, which contains its U-mate. Its third vertex cannot lie in V: the opposite V-endpoint would receive a glue on a separating line, impossible at its $f = 9$ cost. Thus the third vertex is w or q, and both V-endpoints lie in the complementary half. Both receive a nonzero glue, forcing their dense lines into this same split. Those two dense lines in turn force the other U-endpoint's dense line into it. Consequently all four D endpoints are dense in one split, with U and

V on opposite sides. The vertex w receives distinct glues from the two D endpoints opposite it. The exact checker covers every matching, choice of D edges and choice of w, with both possible splits.

Write $f(w) = g + 2s + 2N = 11$. Its glue count is odd, so $g \geq 2$ implies $g \geq 3$. Its balanced N count is positive and even, and $s \geq 1$. Therefore $N = 2$ and $s \leq 2$. Both D blocks still require mass 2, all four D endpoints contribute zero, and q has nonnegative stars of total mass 1. The w contribution to each D block must consequently be at least 1. At most two w-stars are available, so there must be one positive star in each block and none elsewhere. Their total is 1. Together with q's total 1 they cannot supply the required 4. No positivity premise was imposed on w before the two block equations forced this conclusion.

All $Z = 0$ cases are excluded.

3. Z present: cancellation costs exclude every budget. A sole Z needs $C \geq 6$ and is impossible since $\Pi \geq 25$. Thus $z \geq 2, C \geq 2, \Pi \leq 27$ and $h \leq 4$.

Suppose $A = 0$. All three M-arrays cancel and $h \geq 3$. The M-edge containing q has just one active endpoint, whose class must equal 2 and whose cost is at least 11. Other active costs are at least 8 and q costs 5, so $\Pi \geq 24 + h$. For $h = 4$ this already exceeds 27. For $h = 3$ exactly one helper must cancel each M-array, with no other helper. Every D is 1 and every Z is 2. If there are one D and two Z, their supports have four distinct active endpoints, costing at least $18 + 22$, while the fifth costs 9 and q costs 5; hence $\Pi \geq 30$. Three Z give $\Pi \geq 31$. Thus $A \geq 1, \Pi \leq 26, h \leq 3$.

For $h = 2$ there are exactly two Z. If attached at the same center, their cost 13, the four other active costs 36 and q's5 give $\Pi \geq 29$. Otherwise the two Z centers cost at least 8 each and the other three active centers cost 9 each, giving $\Pi \geq 26$. $R = 29$ forces $\Pi = 26, A = 1, C = 2, X = 0$. Two M-arrays would have to cancel, so the Z are in different M-blocks and both equal 2. Their cost rises from 8 to at least 11 each, contradicting $\Pi = 26$.

For $h = 3$, three Z can reach at most three active vertices; at least two active vertices are helper-free. Their minimal total gives

$$\Pi \geq \text{ceil } ((3 \cdot 8 + 2 \cdot 9 + 5)/2) + 3 = 27$$

The only other possibility is one D and two Z. They reach at most four active vertices. $\Pi = 26$ gives doubled line budget 46, attained only with four distinct helper-bearing $f = 8$ vertices, one helper-free $f = 9$ vertex, and $qf = 5$. Every nonzero class is $1/2$, so every block has β at most 1. No M-array cancels, contradicting the necessary $A = 1$. More collisions or helper-free vertices increase the cost. This excludes every Z case.

S15.6. **One matching and one outside physical-helper block.**
Scope. Matching $R \le 29$, two active centers, $d = 0$, two Z at distinct attachments in distinct blocks, one in M and one outside.
Obstruction. Split by flow in Z-free blocks. Projection settles the first branch; three-block flow and signed divergence settle the second.
Conclusion. Both branches are excluded.

Assume matching corrections, $R \le 29$, $m = 2$, $d = 0$, $z = 2$, and distinct physical attachments in distinct blocks. The matching helper has coefficient $a \neq 0$ at u, source $e = \{u, p\}$, and group M. The outside helper has coefficient $b \neq 0$ at $w \neq u$, source $U' = \{w, w'\}$, and group $G \neq M$. The two untouched matching arrays and the outside helper array require $A \ge 3$; if $A = 3$, then $a = 2$. We separate the cases according to nonzero flow in Z-free blocks. Idle loops are retained throughout.

No nonzero flow in a Z-free block. Write $c_i = \mathbf{1}_{W_i} - \mathbf{1}_{V_i}$ for the two block vectors. Projection is

$$2\delta_1 c_1 + 2\delta_2 c_2 = a(e_p - e_u) + b(e_{w'} - e_w). \tag{S15.1}$$

This equation includes arbitrary flow values and every idle-loop placement.
Equal magnitudes. If the two sources intersect, both block vectors vanish at a shared vertex, where the right side gives $\pm a \pm b = 0$. Thus $|a| = |b|$. For equal sources, distinct attachments give $w = p$ and $a = b$; the two nonparallel block vectors then force both flows to vanish. For disjoint sources, e cannot be a side of G: the left side is constant on e, while its right side is $(-a, a)$. Similarly U' cannot be an M-edge. In the remaining configuration, e is absent from G and U' crosses the two other M-edges. The two coordinates outside $e \cup U'$ give $|\delta_1| = |\delta_2|$. Differences at the endpoints of e and U' give $|\delta_2| = |a|/2$ and $|\delta_1| = |b|/2$. Hence again $|a| = |b|$.
Cost reduction. The line sum is $2\Pi - 4$, each active cost is at least eight, and each inactive cost at least five. If $\Pi \le 22$, either active center has cost at most twelve, and any inactive center at most nine. Neither active center has a nonzero source separator, and all inactive B entries are zero or one. If $A = 3$, cancellation forces $a = 2$, then $|b| = 2$, so both active costs are at least eleven and $\Pi \ge 23$. At $\Pi = 23, A = 3$, equality gives costs $(11, 11; 5, 5, 5, 5)$ and the same separator and inactive-row conclusions.
The four physical source cases. At these costs, $B_{up} = 1 - 2a$ and $B_{ww'} = 1 - 2b$.

(1) If $e = U'$, projection gives $a = b$ and zero flows. The source equation requires $H^0_e = 2 - 2a$, whereas its two directed entries give $H^0_e = 1 - 2a$.
(2) If e is a side of G, antisymmetric projection on e forces $a = 0$, contrary to its being a selected helper.
(3) If e is absent from G and $w \neq p$, the mate p is inactive. The source equation gives $H^0_e = 2 - a$, so $B_{pu} = 3$, contradicting its inactive value in $\{0, 1\}$.

(4) If e is absent from G and $w = p$, then w' is inactive and U' is absent from M. Its source equation gives $H^0_{U'} = -b$, so $B_{w'w} = 2H^0_{U'} - B_{ww'} = -1$, again impossible.

These cases exhaust the location of e relative to G. The proof uses the full-column source, projection and separating-sum identities, on which every mixed column has zero coefficient. It therefore applies with arbitrary X and with idle loops. No bound on a general active cost-eleven or cost-twelve B row is used.

The two affine corners. Only $\Pi \geq 23$ remains. Since $A \geq 3$ and $C \geq 2$, the exclusions above leave exactly

$$(\Pi, A, C, X) = (23, 4, 2, 0), \qquad (24, 3, 2, 0).$$

Each helper block needs a loop, exhausting $C = 2$. Without cancellation, four nonzero source arrays consume $A = 4$; with cancellation, the three mandatory arrays consume $A = 3$. An additional zero-total array needs two further columns and cannot fit. Thus all alternatings lie in the three M source arrays and the outside helper's source array. For each physical placement, the containing set is precisely

$$270 \text{ private} + 120N + 2Z + 16A + 18 \text{ loops} = 426.$$

The seventy exact row certificates in S10.4 cover these sets, including all twenty-eight placements surviving projection at $C = 2$. This is the only use of those certificates in the present component; the cases with $X > 0$ have already been excluded analytically.

A nonzero flow in a Z-free block. Each helper block needs one loop, and a Z-free flowing block needs two. Thus $C \geq 4$. The two active centers give $\Pi \geq 20$. If $A = 3$, then $a = 2$ and the active costs give $\Pi \geq 22$. Together with $R \leq 29$, these leave

C	A	Π	
4	3	22	
4	4 or 5	20 or 21	$(\Pi + A \leq 25)$
5	4	20.	

The unused support in each row is $X = 29 - \Pi - A - C$ when $R = 29$; smaller retained budgets only tighten the inequalities. There is exactly one Z-free flowing block. In particular there is no unlisted second such block or extra idle block: either would need another two loops. Of the sixteen necessary positive-X records at $R \leq 29$, fifteen have $C \leq 3$ and hence belong to the preceding no-third-flow proof. The only record with $C \geq 4$ is $(20, 4, 4, 1)$, handled below if a third flow is present and by the preceding proof otherwise. This accounts for every positive-X budget without enlarging a restricted affine space.

Four loops. Each helper block has exactly one L column. In projection, its net helper/loop contribution is its coefficient times a vector with six entries ± 1. The residual is a multiple of the third block vector, which has two zero coordinates. At either zero coordinate cancellation forces $|a| = |b|$. The $A = 3$ alternative would give $|a| = |b| = 2$ and $\Pi \geq 23$, contradicting its budget. Thus $A \geq 4$ and $\Pi \leq 21$. The local class lists now force common magnitude $1/2$ or one.

Magnitude one forces $a = b = 1$, two active costs nine, and four inactive costs five. Therefore $(\Pi, A, C, X) = (21, 4, 4, 0)$. The eighty-four literal systems of S10.4 give sixty-eight exact affine contradictions. The other sixteen force either an inactive reverse entry -1 or an inactive–inactive value $H^0 = 2$, contradicting the respective bounds $B \in \{0, 1\}$ and $H^0 \leq 1$. These sixteen are excluded by costs and directed entries, not by an affine-inconsistency assertion.

For magnitude $1/2$, at most one helper is negative. If both are positive, both active B rows vanish and $H^0 \leq 1$ everywhere. Each one-loop helper contributes net $1/2$ at its source and at one side. The matching block contributes total one on M, and the outside block at most $1/2$. Writing $D_e = H^0_e + \mathrm{net}_Z(e) - 2$ for $e \in M$, every D_e is nonpositive and at least two are strictly negative: equality requires $\mathrm{net}_Z(e) \geq 1$, possible at at most one node. One additional block flow has only one negative divergence node and cannot supply these demands.

If one helper is negative, its center costs ten and has two main marginals $-1/2, -1/2$. Its source entry is two, but every outside-prefix entry is at most one. The other center is glue-free of cost eight, and every inactive center is F5. Hence $H^0 \leq 1$ except possibly at the negative helper source, where $H^0 \leq 3/2$. The net negative contribution $-1/2$ there lowers the total to at most one. Elsewhere that helper's contributions are nonpositive, and the positive helper adds at most $1/2$. Thus $D_e < 0$ on all three matching nodes, again impossible. These demand arguments annihilate mixed columns and include the $(20, 4, 4, 1)$ boundary.

Five loops. Here $(\Pi, A, C, X) = (20, 4, 5, 0)$, so $a = b = 1/2$, both active B rows vanish, all inactive rows are binary, and $H^0 \leq 1$. If both helper blocks have one L, the preceding positive-half demand argument applies; three loops in the third block still provide only one flow and one negative divergence node. If the outside block has two loops and the matching block one, its flow may be arbitrary, but its group contains at most one M-edge. The other two matching edges receive at most $1/2$ from the matching helper and have strictly negative demand. If the matching block has two loops and the outside block one, the matching source has $H^0 \leq 1/2$ and receives at most $1/2 + 1/2$ from the helpers, so its demand is strictly negative. On the other two matching edges, the matching block's net contributions sum to $1/2$ independently of its flow. Their H^0 values sum to at most two and the outside block adds at most $1/2$ in total. Their combined demand is therefore

at most $2 + 1/2 + 1/2 - 4 = -1$; at least one is negative. The single extra flow cannot supply these two negatives.

Both flow branches are exhausted. Every identity in the stated one-in/one-out family has $R \geq 30$.

S15.7. **One active center at retained support twenty-nine.**
Scope. Matching $R \leq 29$, one active center, nonzero helpers; normalization gives $d = 0$.
Obstruction. Cover all one- and two-Z records. Complete the final two-class cores at every inactive center and compare alternating block demand with budget.
Conclusion. No such identity; rational class coefficients have no integer restriction.

1. Notation and stated scopes. In this section we label the six vertices $0, \ldots, 5$. Relabel the unique active center as $u = 0$ and the matching as $M = \{01, 23, 45\}$. The matching-rigidity lemma O29, already proved for $R \leq 29$, gives $\Omega = -2e_M$ and $w = 2\mathbf{1}_M$. Use (S4.8)–(S4.9), with $f(x) = g(x) + 2s(x) + 2N(x)$ and $\Gamma(u) = \sum Z - 1/2$. Here $s(x)$ counts actual stars. The normalization gives $d = 0$ because there is only one active center. Each physical Z therefore occupies its own block and determines a distinct nonzero N class. On each of the class's two h-sides, the N coefficients sum to $-Z$. Its minimal realization uses one coefficient $-Z$ on each of two distinct private lines. A balanced pair instead has opposite coefficients on distinct private lines. Classes with different source prefixes share only the half containing u and both source mates. Classes with the same prefix but different groups have different V-positions on every common line.

The actual stars satisfy (S4.1); in vector form $S = l - r\mathbf{1} - \nu$, with $r = \sum l$ and ν the aggregate N vector. Inactive centers have $f \geq 5$, no $f = 6$, and their strict excess over 5 starts at 2. An inactive $f \leq 9$ center has one marginal $-1/2$ and B entries 0 or 1. Its $f = 5$ line is either sparse with one glue $-1/2$, or dense with one positive $+1/2$ and two negative $-1/2$ glues. All its actual stars are nonnegative.

Use (S4.5) for B, H^0, the flow and projection laws (S4.6)–(S4.7), and the physical-block law (S4.4). Here $c_B = \mathbf{1}_W - \mathbf{1}_V$; the contribution of a physical Z to projection is $Z(e_{\text{mate}} - e_{\text{attachment}})$. Mixed columns vanish in all these equations and in V/h. Since $m = 1$, the block value β is its Z coefficient. Every Z block needs a loop; a Z-free flowing block needs two loops. A sole global Z requires $C \geq 6$.

We use the **SZ6** source-divergence lemma only when $C = 6$: if there is exactly one physical Z in the whole identity, its source has zero flow divergence. The cases $C = 7$ and $C = 8$ require their own arguments. The proved two-class **T13** lemma is used only to exclude a value 2 class at $f \leq 13$. The parity argument in item 5 permits the actual star at every position.

2. Two additional glue obstructions.

2.1 Five inactive $f = 5$ centers permit at most one positive active glue. Every nonzero active glue has magnitude $1/2$, forced by its opposite inactive $f = 5$ line. A positive active glue forces that opposite line to be dense, so the opposite center and the other two centers on the active half all have their unique glue lines in that private partition.

Two such positive glues in different partitions would force three inactive centers in each partition. The two three-element sets overlap among the five inactive centers, forcing a second private line at an $f = 5$ center.

For two positive glues in the same active row, both opposite columns are dense. The two other rows each receive two negative $-1/2$ glues, hence are dense too and have positive $+1/2$ glues in the third column. That column would have marginal 1 plus its active entry, one of $1/2, 1, 3/2$, rather than the required $-1/2$. Thus there is at most one positive active glue.

Three consequences will be used below:

- With a sole Z whose $\Gamma = Z - 1/2$ is positive, there is exactly one active $+1/2$ glue and no negative glue, so $Z = 1$.
- At any active center the sum of separating marginals is at most $1/2$.
- If two Z have opposite values, $\Gamma = -1/2$, so there are either one negative glue, or two negative and one positive glue. In particular, $B_{uv} \le 2$ for every vertex v.

The checker confirms all 435 pairs of potential positive active glues: 405 different-partition conflicts and 30 same-partition conflicts.

2.2 One exceptional inactive $f = 7$ center is impossible. If one inactive center is $f = 7$ and every other inactive center is $f = 5$, the $f = 7$ line has glue vector $(-1/2, t, -t)$, up to permutation, where $t \neq 0$ and $|t| \neq 1/2$. Its two unusual glues cross two distinct centers. Only the active center u could be an unusual crossing partner, since every other inactive glue has magnitude $1/2$. This is impossible, regardless of the active coefficient or balanced N. No second or zero-marginal line fits at $f = 7$.

3. Complete helper and budget cover. At most one Z block can lie in M. Its two other arrays remain nonzero; each outside Z creates its own nonzero array. Hence $A \ge z + 1$. For $z \ge 3$, the five inactive minima, two N per class, and an actual star give

$$\sum_x f(x) \ge 25 + 4z + 2, \qquad \Pi \ge \left\lceil \frac{27 + 4z}{2} \right\rceil + z = 14 + 3z,$$
$$C \ge z, R \ge 15 + 5z \ge 30$$

Only $z = 1$ and $z = 2$ remain. The necessary arithmetic envelope contains 40 records with $m = 1$, 20 with $z = 1$ and 20 with $z = 2$. Some are deliberately overinclusive: for example $\Pi = 18, A = 2$ forces a sole matching Z of coefficient 2, whose minimum active cost 11 already gives $\Pi \ge 19$.

Every actual case is inside the following location refinements, including every allowed allocation between loop cost C and mixed-column cost X:

Helpers	Largest possible Π	Minimum A	C minimum
One Z in M, value 2	21	2	6
One Z in M, value not 2	20	3	6
One Z outside M	19	4	6
Two Z, one in M with value 2	24	3	2
Two Z, one in M with value not 2	23	4	2
Two Z outside M	22	5	2

Earlier $R \le 28$ cases remain covered by their proof. The argument below closes all 40 necessary $R = 29$ records directly.

4. One Z: closure including the $C = 7$ boundary. Let a be the coefficient of the sole physical Z, let t be its source mate, and let T be the sum of separating marginals at u. The directed source entry is

$$B_{ut} = 1 - 2a + 2T$$

Suppose $C = 6$ and the inactive source mate is light, so that $B_{tu} \in \{0, 1\}$. Combining SZ6 with the source equation gives

$$T = \begin{cases} (3 - B_{tu})/2 \in \{1, 3/2\}, & \{u, t\} \in M, \\ -(1 + B_{tu})/2, & \{u, t\} \notin M. \end{cases}$$

If no separator is present, then $T = 0$, which would require $B_{tu} = 3$ or $B_{tu} = -1$. Both values contradict lightness. Thus a nonzero separator is necessary, and the active center must have $f \ge 13$.

$\Pi = 18$ *and all value* 2 *cases.* At $\Pi = 18$, the private-cost identity gives $\sum_x f(x) = 2(18 - 1) = 34$. The active cost is 9 and every inactive cost 5; active $f = 8$ would leave the impossible inactive excess 1. The stated $f = 9$ law gives $a = 1$ and a dense main line with two positive glues, contradicting item 2.1. This excludes every $\Pi = 18$ tuple, independently of C or X.

For $a = 2$, active $f \ge 11$. At $\Pi = 19$ active $f = 11$ and all inactive $f = 5$ are forced; $\Gamma = 3/2$ contradicts item 2.1. At $\Pi = 20$, the alternatives are active $f = 11$ with one inactive $f = 7$, active $f = 12$ with excess 1, or active $f = 13$ with all inactive $f = 5$. The first alternative is excluded

by item 2.2; the second would require the impossible inactive excess 1; the third contradicts item 2.1.

At $\Pi = 21$, the support budget forces $A = 2$, $C = 6$, and $X = 0$. Active $f = 11$ or $f = 12$ has no separator and every inactive center is light, so SZ6 excludes it. Active $f = 13$ would leave a sole inactive $f = 7$; $f = 14$ would leave excess 1; $f = 15$ leaves all inactive $f = 5$ and again contradicts $\Gamma = 3/2$. No $C = 7, C = 8$ source-divergence claim is used.

Worked example 1. A canceled array still leaves a private-cost obstruction.

Consider the hypothetical one-center configuration with matching $M = \{01, 23, 45\}$, active center $u = 0$, no D, and exactly one physical Z: it lies in block $(M, 01)$, attaches at 0, and has coefficient 2. Assume $R \leq 29$ and $\Pi = 19$. Balanced N classes and arbitrary signed coefficients remain allowed. We show why these assumptions cannot describe an identity.

First, $\beta = 2$ cancels the source-array total $t_{01}^{M} = -2 + 2 = 0$. The arrays at 23 and 45 still have total -2, so $A \geq 2$. Canceling this one array has not removed the private coefficients needed to support its helper.

Indeed, $\zeta_0 = 2$ and the center-mass law gives $\Gamma(0) = 2 - 1/2 = 3/2$. The one-class cost bound in S13.5, Part H, item 3.1 gives $f(0) \geq 11$. There is one helper, so the private-cost identity gives

$$\sum_{x=0}^{5} f(x) = 2(\Pi - h) = 2(19 - 1) = 36.$$

The five inactive centers cost at least 25 in total. Consequently $f(0) = 11$, and every inactive center has $f = 5$.

Now apply item 2.1: all active glues have coefficient $\pm 1/2$, and at most one is positive. Their sum, which is $\Gamma(0)$, is therefore at most $1/2$. This contradicts $\Gamma(0) = 3/2$. The contradiction is independent of the permitted allocation to C and X; SZ6 is not needed for this example.

Remaining $\Pi = 19$ *cases.* At $C = 6$, active $f \leq 11$ and inactive $f \leq 8$; no separator fits, so SZ6 excludes the case. At $C = 7$ only the noncanceling matching-Z budget $\Pi = 19, A = 3, C = 7, X = 0$ remains. Here $\sum_x f(x) = 2(19 - 1) = 36$, so the active cost has the following four possibilities:

Active f	Cost consequence and exclusion
8	Active $f = 8$ is the glue-free half class, leaving one inactive $f = 8$. Its two-glue vector $(+1/2, -1)$ requires an unusual -1 glue to u, impossible.
9	Active $f = 9$ leaves one inactive $f = 7$, excluded by item 2.2.
10	Active $f = 10$ leaves excess 1, impossible.
11	Active $f = 11$ leaves five inactive $f = 5$ centers.

In the last row, the active center has $f = 11$. Its glue count g is odd and at most 5, and item 2.1 allows at most one positive glue. Three main N columns would occupy three lines and cost at least 12; four main columns would occupy four lines. The only possible N supports are therefore the minimal main pair and that pair together with one balanced pair.

With $g = 1$ the only nonzero class value is $a = 1$. With the minimal pair there are three actual stars: on the positive-glue main line they are $+1/2$ and $-1/2$, and on the other main line $+1$. With four N there is only the actual star 1. With $g = 3$ the only negative nonzero value is $a = -1$ and every glue is negative. A line with three negative glues leaves at least two actual stars after its single $N = +1$, and the other main line another star; split glues require still more. This exceeds the available two-star budget. With $g = 5, a = -2$ cannot fit a main $+2$ in an effective entry of size at most $3/2$. For $a = -1$ only one actual star 1 is available: a main line carrying that star and its $N = +1$ would require an integer-magnitude glue, whereas both star-free main lines would require positive glues. Neither is possible.

The remaining $a = 1, g = 1$ cases have seven or nine distinct blocks charged by positive inactive stars. There are exactly six inactive glue completions: four have seven charged blocks and two have nine. The one possible negative active star can erase at most one of these blocks; the sole helper can cancel at most one more demand. Hence $A \geq 5$, contrary to $A = 3$. If four N columns are present, the sole actual star has coefficient 1, so it cannot erase any of these positively charged blocks. The same lower bound on A therefore holds, with room to spare. This closes $C = 7$ without using SZ6 there.

Noncanceling $\Pi = 20, A = 3, C = 6$. Now $\sum_x f(x) = 2(20 - 1) = 38$. Subtracting the five inactive minima leaves the exhaustive active costs $8, \ldots, 13$, treated below.

Active f	Cost consequence and exclusion
8	$f = 8$: only an inactive $f = 10$ could supply the exceptional source entry 3. Its two F_5 lines would have marginals $-3/2$ and 1, needing glue magnitudes $3/2$ and 1. Their crossing partners are inactive $f = 5$ or the glue-free active center u; neither is possible. Every other inactive source mate is light.
9	$f = 9$: the inactive excess 4 is one $f = 9$ or two $f = 7$ centers. Its active glues have magnitude $1/2$. A sole $f = 9$ line is thereby forced onto the half grid and degenerates to F_5. Each of two $f = 7$ centers needs two unusual partners and has only the other $f = 7$ as a possible partner. Both alternatives fail.

Active f	Cost consequence and exclusion
10	$f = 10$: one inactive $f = 8$ is forced. The exact one-class $f = 10$ list is $a = 1/2, a = -1/2$, or $a = -1$. The first two have an even glue count and integer Γ; one glue -1 plus the other half-magnitude glues gives a half-integer Γ, impossible. For $a = -1$ the main lines have marginals -1 and $-1/2$. Their glue vectors are a single -1 and a dense half-grid vector $(+1/2, -1/2, -1/2)$. The positive dense glue forces all five inactive glue lines into its partition, while the $f = 8$ partner of the -1 glue is already selected in the other main partition. Contradiction.
$11, 12$	$f = 11$ leaves one $f = 7$, excluded by item 2.2; $f = 12$ leaves excess 1.
13	$f = 13$ leaves all inactive $f = 5$. item 2.1 gives $T \leq 1/2$, whereas SZ6 needs $T \geq 1$ on the matching source.

For the $f = 10$ list: three main N cost at least 12; four N at $f = 10$ force no glue and $a = 1/2$. With two N columns, the main-line types $\mathrm{F}_4 + \mathrm{F}_6$ give $a = 1/2$. The types $\mathrm{F}_5 + \mathrm{F}_5$ give $a = -1/2$ when both lines are sparse, and $a = -1$ when one is sparse and the other dense. This retains the negative value -1 case, which cannot be omitted from the cost cover.

All one-Z budgets are excluded.

5. Two Z with $C \leq 3$: the signed matching-edge obstruction. Write the two nonzero Z coefficients as a and b. The two Z blocks already require one loop each. A flowing block without Z would require two additional loops, exceeding $C \leq 3$. Thus all flow is confined to the two helper blocks, and projection at u gives $a + b = 0$.

Equal source prefixes have zero projected right side. Their distinct non-parallel block vectors force both flows zero, which needs two loops per Z block, hence $C \geq 4$. For different source mates t_1, t_2, each other's mate coordinate in projection gives

$$2\delta_1/a \in \{+1, -1\}, 2\delta_2/a \in \{+1, -1\}$$

This is valid even if an idle loop contributes to $C = 3$. Consequently the effective side charges $\beta/2 \pm \delta$ of each Z block are precisely 0 and its signed β. There is just one block with a positive β, and it charges just one side pair positively. Neither helper source can be 23 or 45. At least one of the two inactive M-edges therefore has no positive effective side charge. The flow equation at that source requires $H^0 \geq 2$.

Whenever every inactive center is light, this contradicts $H^0 \leq 1$ on both inactive M-edges. For $\Pi \leq 23$ lightness follows from active $f \geq 13$:

$$f(\text{inactive }) \leq 2(\Pi - 2) - 13 - 4 \cdot 5 \leq 9$$

The sole larger tuple is $\Pi = 24, A = 3, C = 2, X = 0$. $A = 3$ forces the matching Z of coefficient 2 and, by projection, the other $Z = -2$ on a different source. Such a center needs $f \geq 15$. For $f \leq 14$ five main N already use four lines; six N plus a required glue and star cost at least 15. Four main N on three lines are excluded by 8088 glue and actual-star masks with unrestricted rational coefficients, covering coincident and different positions on the common line. No extra N-free line fits. Thus the inactive bound becomes $44 - 15 - 20 = 9$, and the same signed side obstruction applies.

The checker considers all 105 class pairs at u: 15 have equal sources, 60 have inconsistent different-source projections, and 30 have consistent projections. Both amplitude signs give 60 exact inactive-M-edge contradictions. This uses no O15 placement assumption and no positive-helper premise.

6. The two $C = 4$ cases. At $C \geq 4$ the only necessary records are

$$\Pi = 21, A = 3, C = 4, X = 1; \Pi = 21, A = 3, C = 5, X = 0;$$
$$\Pi = 21, A = 4, C = 4, X = 0; \Pi = 22, A = 3, C = 4, X = 0$$

In the first two, active $f = 13$ and matching Z of coefficient 2 contradict T13, regardless of where the remaining flow travels. The last two records are resolved below by counting the physical blocks that require alternating directions. These arguments do not require a three-block restriction or a source-divergence lemma.

$\Pi = 21, A = 4, C = 4$. Active $f = 13$ and five inactive $f = 5$ centers are forced. There are four main N. Five N at equal prefixes cost at least 14, and at different prefixes at least 16; six N plus a star cost at least 14. Four lines cost at least 16, so the main support has either two common lines or one common line and two singleton lines. The active glue count is 1 or 3, with at most one positive glue by item 2.1.

The checker covers each of these three local incidence types. It allows both helper values to be unrestricted rationals. Center mass gives $a + b$ from the specified glues; actual zero-star equations determine the finite coefficient roots. Generic free coefficient families are also checked, and none meets this cost. The three incidence types yield 8, 6, and 1 local forms, respectively, for a total of 15.

Embed these forms in every literal minimal main-pair placement with one block in M. For each embedding, complete each inactive line by one of its six proved F_5 forms. Nonzero entries force its partition, and every other line at that center is zero. This gives 7568 completions of analytically forced minimum private cores, not a census of arbitrary PM2 supports. Each completion satisfies all 180 literal V-equations, all 180 literal h-equations, and all six center-mass equations. Their block demands require **at least five alternating directions**. Hence no $A = 4$ completion exists, regardless of flow or mixed columns.

$\Pi = 22, A = 3, C = 4$. One matching Z equals 2. Active $f = 13$ is excluded by T13; $f = 14$ leaves the impossible inactive excess 1. Thus active $f = 15$ and five inactive $f = 5$ are forced.

For five main N with different prefixes, four lines already cost 16. With equal prefixes there are two double-N lines and one singleton line, with minimum costs $5, 5, 4$. If $g = 1$, the remaining budget permits two actual stars. The sole glue cannot serve both lines containing two N columns: one such line would be glue-free and cost 8, raising the total to at least 17. If $g = 3$, the budget permits only one actual star. There is then an isolated main coefficient $N = -2$ at a V-cell, because a class with three main N columns has one h-side containing a single column. Equal-prefix classes have different positions, so that cell cannot be canceled by the other class. But at most one positive glue gives every effective entry $\geq -1/2$; subtracting the sole actual star 1 leaves aggregate $N \geq -3/2$. Contradiction.

Six main N with equal prefixes need three double-N lines of minimum cost 5 each and leave no actual star at cost 15; four lines cost too much. Different prefixes need at least five lines. The remaining six-N option is a minimal main pair for each class plus one balanced pair. Cost 15 forces one glue and one actual star 1. The glue is $\pm 1/2$, so the other helper is -2 or -1 and all four main N are integers. The two half-integer aggregate entries on the sole glue line cannot be supplied by a balanced pair on distinct lines. This argument permits the actual star anywhere. Seven N plus the required star already exceed 15.

It remains to consider four main N columns, with active glue count $g = 1, 3$, or 5. For the same three local incidence types the fixed-value 2 local calculation gives six forms, all with the other coefficient -2 and a common line on which the main N coefficients cancel in the same cell. Completing the five inactive F_5 centers gives 7136 forced cores. Every core satisfies the V- and h-equations and all center equations; its physical-block demands require **at least eight alternating directions**, far above $A = 3$. No flow geometry is used, so an additional flowing block does not help.

This excludes every two-Z record.

S15.8. **Three active centers with physical helpers.**
Scope. Matching $R \leq 29$, three active centers, $Z > 0$.
Obstruction. Helper and cancellation costs leave a short list. The signed B_{11} bound controls outside demand; the final mixed-location pair is treated directly.
Conclusion. No such three-center identity; the sole-Z lemmas are used only in their proved scope.

1. Equations, stated local facts, and exact scope. Label the vertices $0, \ldots, 5$, write $M = \{01, 23, 45\}$, and let K be the three active centers, namely the vertices with a selected N column. Write $h = d+z$ for the number of physical helpers. Use (S4.8)–(S4.9), with $f = g+2s+2N$ and the marginal law (S4.2). Here s counts actual stars, whose coefficients sum to 1 at each center. For a

block $B = (G, U = \{u, v\})$, endpoint classes are $\zeta_u = D + Z_u, \zeta_v = D + Z_v$, and $\beta = \zeta_u + \zeta_v$. Alternating array totals are

$t_U^G = -2 \cdot 1[G = M] + \beta_B$. Use (S4.5), the flow and projection laws (S4.6)–(S4.7), and the actual-star equation (S4.1). Here $w = 2\mathbf{1}_M$ and $c_B = \mathbf{1}_W - \mathbf{1}_V$; a physical Z contributes $Z(e_{\text{mate}} - e_{\text{attachment}})$ to projection. An alternating array with zero total, if nonempty, needs at least two columns; an array with nonzero total needs at least one. Every source of a surviving D lies entirely in K; its two endpoint classes are nonzero after the normalization. Every nonzero Z has an attachment in K.

Choose the representative that minimizes (R, z, d) lexicographically. It has no equal nonzero endpoint Z pair and satisfies $C \geq z$. A Z-free block with nonzero flow costs at least two loops. A block with one Z has $|\delta| = |Z|/2$ when it has just one loop. Idle loops are included in C and cannot create an additional flowing block for free. For a sole global pair of endpoint Z columns in one block, the earlier one-block lemma applies when $m = 3$, $A \geq 2$, and $\Pi \geq 22$. These bounds give $C \leq 5$; the lemma then forces the two Z coefficients to be equal, contrary to the chosen normal form. We use that result only in this stated scope. The $z = 1$ branch is excluded by Lemma SL10.

Inactive centers cost at least 5, have no $f = 6$, and their strict excess starts at 2. At $f \leq 9$, an inactive center has a single nonzero marginal, equal to $-1/2$, and all its directed B entries belong to $\{0, 1\}$. Active centers cost at least 8; a helper-free active center costs at least 9. At $f = 8$ an active center is glue-free and has a single nonzero class $1/2$. At $f = 9$, a nonzero class is 1; all glues have magnitude $1/2$. Helper-free active $f = 9$ has B in $\{0, 1\}$.

Two nonzero classes cost at least 13. Three cost at least 14. The proved T13 fact excludes a value 2 class at $f \leq 13$ if another nonzero class is present. In its present uses the added value 2 prefix differs from the incident D prefix, or the value 2 obstruction is applied as previously proved. The argument uses this proved class bound; it does not require a separate claim about an isolated shared F_7 line.

For a sole nonzero class at $f \leq 10$ the signed value list is $\{1/2, 1, -1/2, -1\}$; exact $f = 10$ has no value 1. A center of such a class has $B \leq 2$ when queried outside its source prefix. Nonzero marginal separation from that prefix costs at least 13, so if $f \leq 12$,

$$B_{uv} = 1 - 2a,$$

where u is the center, v its source mate, and a the class value. Helper-free active $f \leq 10$ has B in $[-1, 2]$. This bound follows from the balanced N law; its hypotheses allow an active center. A sole value 2 class costs at least 11. Cancellation of an M-array with two active source endpoints requires the two endpoint costs to sum to at least 18. Equality forces both endpoint values to be 1, with cost $f = 9$ at each endpoint. With one active source

endpoint, cancellation needs its lone Z of coefficient 2 and $f \geq 11$. All these local facts include balanced N and arbitrary signs.

2. Two small lemmas.

2.1 Scoped balanced B_{11} *and half-grid parity.* Suppose a helper-free active center w has $f \leq 11$, the other two active centers are glue-free, and all three inactive centers are F_5. Every glue at w must cross an inactive F_5 line, so its coefficient is $\pm 1/2$. Since w is helper-free, $\Gamma(w) = -1/2$. Under these hypotheses, **every directed** B **entry at** w **is at most** 2.

It remains to prove the boundary case $f = 11$. Every balanced class contains an even number of N columns: each h-side has exactly two possible columns, and its zero class sum requires both to be present or both absent. The two columns on that side occupy different private lines. The nonzero value of Γ requires a glue, and center mass 1 requires an actual star. Thus the number of N columns is either 2 or 4.

If there are four N columns, the cost identity $11 = g + 2s + 2N$ forces one glue and one actual star, so $B \leq 1$. If there are two N columns, then $g = 1, 3$, or 5. For $g = 1$, there is at most one negative glue of coefficient $-1/2$, giving $B \leq 1$. For $g = 3$, there are at most two such negative glues, giving $B \leq 2$.

For $g = 5$, the budget leaves one actual star, necessarily of coefficient 1. Every glue lies on one of the two N lines, since another nonempty line would cost at least $5 + 4 + 4 = 13$. The glue counts on the two lines must therefore be 3 and 2. A line with two glues of coefficient $\pm 1/2$, no actual star, and only one nonzero aggregate N entry cannot satisfy the cell equations. Hence the actual star lies on the two-glue line. The three-glue line is star-free, and its F_5 law with one N column forces the N amplitude to have magnitude 1.

On the two-glue line, the marginal r is an integer, whereas the effective-star entries at the two nonzero glue positions are half-integers that are not integers. The integer N amplitude and the actual star of coefficient 1 cannot supply those entries. This excludes $g = 5$ and proves the bound.

A direct exact control checks all 3240 masks: the missing glue, two positive glue positions, both N positions, and the location of the actual star. The N amplitude is reconstructed from one cell equation as an unrestricted rational root, and every other cell is checked. There are no survivors.

We also use the simpler parity consequence of half-grid glues:

$$f \equiv g \equiv 2\Gamma \pmod 2$$

Thus a sole class $3/2$ at $f = 11$, a sole class $-1/2$ at $f = 11$, or a sole class -1 at $f = 10$ is impossible when all its crossing glues are forced onto that half grid. Each application below first proves that all other possible crossing centers are either glue-free or have half-magnitude glues.

2.2 A matching-edge aggregate for the half-Z equality cores. If two Z endpoints are $f = 8$, the third active center is helper-free $f = 9$, and all three inactive centers are $f = 5$, then

$$\sum_{e \in M} H_e^0 \leq 2$$

The two $f = 8$ centers have B rows zero; the other four centers contribute at most 1 each to their own matching mate. Summing H^0 counts each of these six directed entries with factor $1/2$.

Distinct perfect matchings share at most one edge. Consider a block whose group differs from M, with one Z of coefficient $1/2$ and one loop. Then $|\delta| = 1/4$. Its total helper contribution minus flow divergence on M is at most $1/2$: if its source is in M the helper term is $1/2$ and divergence there is zero; if one side is in M the helper term is $1/4$ and flow correction at most $1/4$; otherwise it is 0. Two such outside blocks therefore give

$$6 = \sum_{e \in M} H_e^0 + \text{helper contribution} - \text{flow divergence} \leq 2 + 1 = 3,$$

an exact contradiction.

If instead both half-Z blocks are in M, each contributes total 1 on M, with matching-summed divergence zero. Nonzero antisymmetric projection forces an outside Z-free flowing block. At $C = 4$ this is the only extra flow block: the two Z blocks each consume one loop, and it consumes two. On each half-Z source M-edge the right-side endpoint difference has absolute value 1, while all M flow vectors are constant there. Hence

$$2|\delta_{\text{out}}| \cdot |c(u) - c(\text{mate})| = 1,$$
$$|\delta_{\text{out}}| \in \{1/4, 1/2\}$$

An outside block contributes divergence on at most one M-edge, so

$$6 \leq 2 + 1 + 1 + 1/2 = 9/2 < 6,$$

again impossible. The exact projection check covers all 504 source, endpoint and outside-block configurations. 468 are inconsistent; the 36 compatible six-coordinate systems force the stated bounded flow. The aggregate works even if extra idle loops are admitted before the cost equality is established. It does not assume star positivity at unrestricted active centers.

Worked example 2. Projection can hold while matching content fails.

Continue with $M = \{01, 23, 45\}$. Consider a hypothetical identity with active centers $K = \{0, 2, 5\}$, no D, and exactly two physical Z columns. Specify their blocks by

	G	U	(V, W)	attachment
B_1	$\{01, 24, 35\}$	01	$(24, 35)$	0
B_2	$\{02, 13, 45\}$	45	$(02, 13)$	5

Both block groups differ from M, although both source edges belong to M. This distinction matters when counting alternating arrays. Assume $R \le 29$, with arbitrary signed rational coefficients and balanced N classes allowed.

The three arrays belonging to group M remain nonzero, and each outside helper block adds a nonzero array. Thus $A \ge 5$. Each helper block needs a loop, so $C \ge 2$. The two helper endpoints cost at least 8 each; the helper-free active center costs at least 9; and the three inactive centers cost at least 5 each. Hence

$$\Pi \ge \frac{8 + 8 + 9 + 3 \cdot 5}{2} + 2 = 22.$$

Since $22 + 5 + 2 = 29$, equality is forced throughout: $\Pi = 22$, $A = 5$, $C = 2$, and $X = 0$. Both helper coefficients are $1/2$; the two helper endpoints are glue-free; the other active center has $f = 9$; and all inactive centers have $f = 5$. There is one loop in each helper block and no other flow, with $|\delta_1| = |\delta_2| = 1/4$.

These data need not fail the projected equations. With the displayed ordering of V, W, choose $\delta_1 = -1/4$ and $\delta_2 = 1/4$. Writing e_i for the i-th coordinate vector,

$$c_1 = e_3 + e_5 - e_2 - e_4, \qquad c_2 = e_1 + e_3 - e_0 - e_2,$$
$$2\delta_1 c_1 + 2\delta_2 c_2 = \tfrac{1}{2}(e_1 - e_0) + \tfrac{1}{2}(e_4 - e_5).$$

This is exactly the required projection of the two Z columns. It is only a solution of the projected equations, not a PM2 identity.

To see the remaining obstruction, sum the full flow law over the three edges of M. The two glue-free centers contribute zero directed B entries. Each of the other four centers contributes at most 1 toward its matching mate, giving $\sum_{e \in M} H^0_e \le 2$. Each helper contributes $1/2$ at its matching source, and neither side of either block lies in M. The total helper contribution on M is therefore 1, and the summed flow divergence is zero. The full law would require

$$6 = \sum_{e \in M} w_e = \sum_{e \in M} H^0_e + 1 \le 3,$$

a contradiction. The bounds on the directed entries follow from the forced local costs; no positivity assumption on general active stars has been introduced.

3. $A = 0$, repeated D and large helper counts.

$A = 0$. All three M-arrays must cancel. K must meet each M-edge exactly once, because otherwise one M-array has no possible active helper endpoint. Every cancellation is therefore a lone M-Z of coefficient 2. The active costs are at least 33, inactive costs 15, and $h \geq z \geq 3$. Thus

$$\Pi \geq 24 + h, R \geq 24 + h + z \geq 30$$

So $A \geq 1$.

$d \geq 2$. For $d \geq 3$, the source multigraph on three vertices has at least two vertices of D degree at least 2. Each has at least two distinct nonzero classes, so the active sum is at least $13 + 13 + 8 = 34$. The inactive cost sum is at least 15, giving $\Pi \geq 25 + h \geq 30$ because $h = d + z \geq 5$. This uses no Z-free degree lemma.

For $d = 2$ with repeated source, the same 34 bound gives $\Pi \geq 25 + h \geq 29$, already impossible with $C \geq z \geq 2$.

For $d = 2$ with different sources, the D graph is a path. Its shared center costs at least 13 and the leaves at least 8, giving $\Pi \geq 22 + h$. If $z \geq 3, R \geq 24 + 2z \geq 30$. If $z = 2, \Pi \geq 26, C \geq 2$ forces $A \leq 1$, hence $A = 1$. Two canceled M-arrays include one with only one active endpoint. Its value 2 class is additional to that endpoint's incident D classes, since the M-mate is outside K. At the shared center three classes give $f \geq 14$ and $\Pi \geq 23 + h = 27$; at a leaf T13 gives $f \geq 14$ while the shared center remains at least 13, a stronger bound. Either yields $R \geq 30$. Thus $d \leq 1$.

$A = 1$ *and* $z \geq 4$. Two M-arrays cancel. If K meets all three M-edges once, the active sum is at least $11+11+8 = 30$, giving $\Pi \geq 23+h$. If K contains one full M-edge U and one endpoint of another V, the U cancellation and lone V-Z of coefficient 2 give active sum at least $18 + 11 = 29$, giving $\Pi \geq 22 + h$.

With $d \leq 1$ and $z \geq 4$, or $d = 1, z = 3$, these bounds and $C \geq z$ already exceed 29. For $h = 2$ the helpers are two Z of coefficient 2 in distinct M-blocks and $\Pi \geq 25$, hence $C \leq 3$. For $h = 3$ the two cancellations consume at least two helpers. An outside block with $\beta = 0$ needs at least two more helpers, and an outside block with $\beta \neq 0$ would need an additional alternating. Thus all helpers are in M and $\Pi \geq 25$, again $C \leq 3$. A Z-free outside flow block would need two loops in addition to the Z-block loops, so none fits.

There is an unpaired M-Z: at $z = 2$ the sole-pair-in-one-block exclusion leaves distinct blocks; at $z = 3$ the odd number leaves a lone endpoint. Its antisymmetric right-side projection cannot be supplied by flow vectors constant on every M-edge. This excludes $A = 1$.

Now $A \geq 2$ and

$$R \geq (20 + h) + 2 + z = 22 + d + 2z$$

This inequality excludes every case with $z \geq 4$. Only $d = 0, z = 2$; $d = 0, z = 3$; $d = 1, z = 2$; and $d = 1, z = 3$ remain. At $d = 1, z = 3$ the sole possible tuple is $\Pi = 24, A = 2, C = 3, X = 0$. The three active costs sum to at most 25. No active center can therefore carry more than one nonzero class, and every nonzero class has value $1/2$ or 1. A crossing cancellation needs a value 2, while an internal cancellation requires two $f = 9$ centers and the third costs 8, exceeding 25. Thus $A \geq 3$, a contradiction. We have reduced to the three families below.

4. Three Z, no D. The bounds $A \geq 2$ and $C \geq 3$ force $\Pi \leq 24$. After subtracting the three inactive minima from $\sum_x f(x) = 2(\Pi - 3)$, the active costs sum to at most 27. A multiclass center would cost 13 plus $8 + 8 = 29$, so the three Z attach at distinct vertices, one class each.

At $\Pi = 23$ the active sum is at most 25: every class is $1/2$ or 1, every Z-block β is positive, and no M-array cancels. Hence $A \geq 3$, and only $\Pi = 23, A = 3, C = 3, X = 0$ can fit. There is no outside nonzero β, so all helpers are in M. Three loops are already required by the three Z; all flow is confined to M, and the unavoidable unpaired Z contradicts projection.

At $\Pi = 24$ only $A = 2, C = 3, X = 0$ fits. Two M cancellations cost at least 29 active units (18+11 for one internal plus one crossing cancellation, or $11+11+8$ for two crossing ones), exceeding 27. Exactly one cancels; the two other M-arrays consume both alternatings. Any outside helper block must have $\beta = 0$. With $D = 0$ this would use an opposite endpoint Z pair, whose joint cost is at least 18: one endpoint costs at least 10, since $f = 8, f = 9$ nonzero classes are positive, and the other at least 8. The remaining Z must be the sole canceled M-Z of coefficient 2 and cost 11. Their active costs would sum to at least 29, exceeding the available 27. Therefore all helpers are in M. $C = 3$ leaves no outside flow, and an unpaired M-Z gives the same projection contradiction.

Thus all $d = 0, z = 3$ cost records are excluded.

5. One D and two Z. After $A \geq 2, C \geq 2$, only $\Pi = 23, 24$ or 25 is possible. The Z occupy distinct blocks by the one-block result.

$\Pi = 23$*: the* $A = 4$ *boundary.* The active sum is at most 25, so every active center has one nonzero class. Exactly one Z shares a D endpoint class; the other Z attaches at the third center. Otherwise there is a multiclass vertex, or both Z occupy the D block, already excluded. The D endpoint values are unequal, because their difference is the nonzero shared Z. They must therefore be $1/2$ and 1 at costs $8, 9$; the third Z is $1/2$ at cost 8. All inactive centers are $f = 5$. The two helper blocks have $\beta = 3/2$ and $\beta = 1/2$, and no M-array cancels.

At $A = 3$ all helpers are in M; $C \leq 3$ confines flow to M and gives the unpaired-Z projection contradiction. At $A = 4$ only $C = 2$ fits, and at most one helper block is outside M.

If the D/Z block is in M and the third Z outside, let e be the source of D. The edge e cannot be the source of the outside block: that block's attachment is the third active center, which is not an endpoint of D. If e is a side of the outside group, both flow vectors are constant on e while the shared M-Z gives nonzero antisymmetric projection; the other source is disjoint from e. If e is absent from that group, its equation requires $H^0_e = 2-\beta$ but the two nonseparating endpoint rows give $H^0_e = 1-\beta$. Both are contradictions.

If the D/Z block is outside and the third Z in M, let $e = \{w, p\}$ be the third Z's source. Again, e cannot be the source of D. If e is a side of that group, projection is impossible. If it is absent, the source equation forces $B_{pw} = 3$. However, p is either an inactive center with $f = 5$ or a D endpoint with $f = 8$ or $f = 9$. In the latter case, w lies outside that endpoint's class prefix. The stated local bounds therefore give $B_{pw} \le 2$, a contradiction. This excludes the $\Pi = 23, A = 4$ boundary too.

$\Pi = 24$*: cancellation and the signed* $\beta = 0$ *boundary.* The active sum is at most 27, so every center still has one class. The same D/Z sharing pattern applies. $A = 2$ needs an M cancellation. A sole third M-Z of coefficient 2 would cost 11, leaving joint D endpoint cost at most 16; both would be $f = 8$ of value $1/2$, making the shared Z zero. Cancellation at the D/Z block leaves joint endpoint cost at most 19. The low-cost possibilities are two costs $f = 9$ with values $1, 1$, again making the shared Z zero; or costs $f = 8$ and $f = 11$ with values $1/2$ and $3/2$. The latter forces the other active vertex glue-free and all inactive vertices $f = 5$, so half-grid parity excludes class $3/2$ at $f = 11$. There is no M cancellation. Hence $A \ge 3$.

If $A = 3$, the budget forces $C = 2$. If every helper block belongs to group M, projection is impossible. Otherwise the third Z must be in M and the D/Z outside block must have $\beta = 0$; any nonzero outside β would add a fourth alternating. Its endpoint classes are opposite. An opposite pair costs at least 18, so all inactive centers are $f = 5$ and the active sum is exactly 27. Let u, v be the two D endpoints and w the third active center. Up to interchanging u and v, the only signed low-cost patterns are

$$\big(f(u), f(v), f(w)\big) \in \{(8, 10, 9), (8, 11, 8), (9, 10, 8)\}.$$

The latter two have opposite classes $1/2, -1/2$ or $1, -1$. The other endpoint has half-magnitude glues or none; the third is glue-free; every inactive line is F_5. Thus half-grid parity excludes the negative $f = 11$ or $f = 10$ endpoint respectively. The first pattern has values $1/2, -1/2, 1$. Let e be the source of the third Z, whose block belongs to M. Relative to the outside group, there are three possibilities. It cannot be the source, because the third center is not a D endpoint. If it is a side, the antisymmetric projection cannot be supplied. If it is absent, the source equation forces a directed B entry to equal 3. That entry belongs either to an inactive center or to a D endpoint

of cost at most 10, queried outside its class prefix; in both cases it is at most 2. This closes $\Pi = 24$.

$\Pi = 25$*: allow the multiclass equality, then remove it.* Only $A = 2, C = 2, X = 0$ fits. The active sum is at most 29. If a multiclass vertex occurs, its cost is 13 and the others are $f = 8$, with all inactive $f = 5$. No Z can share the D block in this case: an extra class at a D endpoint would leave a helper-free third attachment and active cost at least $13+8+9 = 30$. Therefore the lone D has coefficient $1/2$ from an $f = 8$ endpoint, and $\beta = 1$. Any M-Z of coefficient 2 at an $f = 8$ vertex is impossible; at the $f = 13$ multiclass vertex it is excluded by T13. There is no M cancellation, contrary to $A = 2$.

Otherwise there is one class at each vertex, with one Z sharing the D block and the other at the third vertex. If the D block cancels an M array, $A = 2$ forces the third Z into M too; all flow is in M at $C = 2$ and projection fails. If the third M-Z of coefficient 2 cancels, an inside-M D block has the same projection obstruction. An outside D/Z block must have $\beta = 0$, since the other two M-arrays consume $A = 2$. The third class costs 11, leaving joint opposite-class endpoint cost at most 18, hence $f = 8, f = 10$ with values $1/2, -1/2$. The third center has no separator. Its source e cannot be the source of D. If e is a side of the outside group, projection fails; if it is absent, the source equation requires $B = 3$ at an inactive center or at a D endpoint of cost at most 10, queried outside its class prefix. Both bounds give $B \leq 2$.

Thus all $d = 1, z = 2$ cost records are excluded.

6. Two Z, no D. If both Z attach at one center, active costs $13, 9, 9$ plus inactive 15 give $\Pi \geq 25$. At most one of their distinct blocks lies in M; even with cancellation the outside block gives $A \geq 3$, so $R \geq 30$. Their attachments are therefore distinct; the third active center is helper-free. A sole Z pair in one block was already excluded. $\Pi \geq 22$.

Both blocks outside M. $A \geq 5$, so only $\Pi = 22, A = 5, C = 2, X = 0$ fits. The two Z centers are glue-free $f = 8$ with $Z = 1/2$; the third active is $f = 9$ and inactive centers $f = 5$. The matching-edge aggregate in item 2.2 gives $6 \leq 3$ for the matching-edge sum, impossible.

Both blocks in M. Unpaired M-Z projection needs an extra outside flow block, so $C \geq 4$. One cancellation gives $\Pi \geq 24, A \geq 2$ and $R \geq 30$; two give $\Pi \geq 25, A \geq 1$ and $R \geq 30$. Without cancellation only $\Pi = 22, A = 3, C = 4, X = 0$ fits. Its private-cost equality forces the two Z coefficients to be $1/2$, so item 2.2 applies and gives $6 \leq 9/2$, impossible.

One block in M and one outside M. Let the Z in group M have coefficient a and attachment u, and let the outside Z have coefficient b and attachment v. If no M-array cancels, then $A \geq 4$, so $\Pi \leq 23$ and $C \leq 3$. The M center has $f \leq 10$; the outside center $f \leq 10$; inactive centers $f \leq 7$; the helper-free

third center $f \le 11$. There is no nonzero separator at either Z center, and every flow remains in the two Z blocks. Let $e = \{u, p\}$ be the M source.

- If e is the outside source, $p = v$. Both endpoint rows are nonseparating; $H_e^0 = 1 - a - b$. Helpers contribute $a + b$, divergence is zero, and the edge weight would be 1 rather than 2.
- If e is a side of the outside group, both flow vectors are constant on e, the outside source is disjoint, and the M-Z antisymmetric projection cannot be supplied.
- If e is absent, its source equation forces $B_{pu} = 3$. If p is inactive, lightness gives $B_{pu} \le 1$. If $p = v$, the bound outside the source prefix gives $B_{pu} \le 2$. If p is the helper-free active center and $f(p) \le 10$, the signed balanced-class bound again gives $B_{pu} \le 2$. Finally, if $f(p) = 11$, the line budget forces both Z centers to have $f = 8$ and every inactive center to have $f = 5$. The two Z centers are therefore glue-free, so all three hypotheses of item 2.1 hold. Its B_{11} bound gives $B_{pu} \le 2$ in this last case as well.

With cancellation $a = 2, A \ge 3$ and $\Pi \ge 24$, so only $\Pi = 24, A = 3, C = 2, X = 0$ fits. The M endpoint has $f \le 12$, the outside endpoint $f \le 9$, the helper-free center $f \le 10$, and every inactive is $f = 5$. The same three possibilities for the source edge—source, side, or absent in the outside group—give the same contradictions. Here the bounds for centers of cost at most 10 suffice; neither SZ6 nor a stronger sole-Z lemma is needed.

This exhausts $d = 0, z = 2$ and hence every identity with $m = 3$ and a nonzero Z at $R \le 29$.

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